Theorem 1 (The Cauchy-Buniakowski-Schwarz Theorem) If \( u, v \in \mathbb{R}^n \), then

\[ |u \cdot v| \leq \|u\| \|v\|. \]

Equality holds exactly when one vector is a scalar multiple of the other.

Proof I. If either \( u = 0 \) or \( v = 0 \), then \( u \cdot v = 0 \) and \( \|u\| \|v\| = 0 \) so equality holds. For the remainder of the proof, we will assume that \( u \) and \( v \) are nonzero vectors.

Let \( \alpha \) and \( \beta \) be arbitrary scalars. Then \( \|\alpha u + \beta v\|^2 \geq 0 \).

Using properties of lengths and dot products,

\[
\|\alpha u + \beta v\|^2 = (\alpha u + \beta v) \cdot (\alpha u + \beta v) = \alpha^2 u \cdot u + \alpha\beta u \cdot v + \beta\alpha v \cdot u + \beta^2 v \cdot v = \alpha^2 \|u\|^2 + 2\alpha\beta u \cdot v + \beta^2 \|v\|^2
\]

Since this holds for all scalars \( \alpha \) and \( \beta \), we are free to choose \( \alpha = \|v\| \) and \( \beta = \mp \|u\| \). Substituting,

\[
\|\alpha u + \beta v\|^2 = \|v\|^2 \|u\|^2 + 2 \|v\| (\mp \|u\|) u \cdot v + (\mp \|u\|)^2 \|v\|^2 = 2 \|v\| \|u\| (\mp \|u\| \mp u \cdot v)
\]

Since \( u \) and \( v \) are nonzero vectors, \( \|u\| > 0 \) and \( \|v\| > 0 \), so \( \|\alpha u + \beta v\|^2 \geq 0 \) is true exactly when \( \|v\| \|u\| \mp u \cdot v \geq 0 \). That is, exactly when \( \|v\| \|u\| \geq \pm u \cdot v \), which is the same as \( \|v\| \|u\| \geq \|u \cdot v \| \).

Note that \( \|v\| \|u\| = |u \cdot v| \) exactly when \( \|\alpha u + \beta v\|^2 = 0 \), which is exactly when \( \alpha u + \beta v = 0 \). Since \( u \) and \( v \) are nonzero vectors, \( \alpha u + \beta v = 0 \) implies either \( \alpha = \beta = 0 \) or else \( u \) and \( v \) are scalar multiples of each other. Since \( \alpha \) and \( \beta \) are nonzero, one vector must be a scalar multiple of the other.
**Proof II.** If either \( \mathbf{u} = \mathbf{0} \) or \( \mathbf{v} = \mathbf{0} \), then \( \mathbf{u} \cdot \mathbf{v} = 0 \) and \( \|\mathbf{u}\| \|\mathbf{v}\| = 0 \) so equality holds. For the remainder of the proof, we will assume that \( \mathbf{u} \) and \( \mathbf{v} \) are nonzero vectors.

First, suppose that \( \mathbf{x} \) and \( \mathbf{y} \) are unit vectors. Using properties of lengths and dot products,

\[
\|\mathbf{x} \pm \mathbf{y}\|^2 = (\mathbf{x} \pm \mathbf{y}) \cdot (\mathbf{x} \pm \mathbf{y}) = \mathbf{x} \cdot \mathbf{x} \pm 2 \mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} = \|\mathbf{x}\|^2 \pm 2 \mathbf{x} \cdot \mathbf{y} + \|\mathbf{y}\|^2 = 1 \pm 2 \mathbf{x} \cdot \mathbf{y} + 1 = 2 (1 \pm \mathbf{x} \cdot \mathbf{y})
\]

Since \( \|\mathbf{x} \pm \mathbf{y}\|^2 \geq 0 \), \( 1 \pm \mathbf{x} \cdot \mathbf{y} \geq 0 \). This is the same as \( 1 \geq \mp \mathbf{x} \cdot \mathbf{y} \). Thus, \( |\mathbf{x} \cdot \mathbf{y}| \leq 1 \). Further, equality holds exactly when \( \mathbf{x} \pm \mathbf{y} = \mathbf{0} \), which means that \( \mathbf{y} = \pm \mathbf{x} \).

Now suppose that \( \mathbf{u} \) and \( \mathbf{v} \) are general nonzero vectors. Then \( \|\mathbf{u}\| > 0 \) and \( \|\mathbf{v}\| > 0 \) so \( \mathbf{x} = \frac{1}{\|\mathbf{u}\|} \mathbf{u} \) and \( \mathbf{y} = \frac{1}{\|\mathbf{v}\|} \mathbf{v} \) are unit vectors. Then

\[
\begin{align*}
|\mathbf{x} \cdot \mathbf{y}| & \leq 1 \\
\left| \left( \frac{1}{\|\mathbf{u}\|} \mathbf{u} \right) \cdot \left( \frac{1}{\|\mathbf{v}\|} \mathbf{v} \right) \right| & \leq 1 \\
\frac{1}{\|\mathbf{u}\|} \cdot \frac{1}{\|\mathbf{v}\|} |\mathbf{u} \cdot \mathbf{v}| & \leq 1 \\
|\mathbf{u} \cdot \mathbf{v}| & \leq \|\mathbf{u}\| \|\mathbf{v}\|
\end{align*}
\]

Equality holds exactly when \( \frac{1}{\|\mathbf{u}\|} \mathbf{u} = \pm \frac{1}{\|\mathbf{v}\|} \mathbf{v} \), which means exactly when one vector is a scalar multiple of the other.
Proof III. Recall that when $a, b, c$ are real numbers with $a \neq 0$, the quadratic function $at^2 + bt + c$ has at most one real root exactly when $b^2 - 4ac \leq 0$. Further, in this case, $at^2 + bt + c = 0$ for some real $t$ occurs exactly when $b^2 - 4ac = 0$.

If either $u = 0$ or $v = 0$, then $\mathbf{u} \cdot \mathbf{v} = 0$ and $\|\mathbf{u}\|\|\mathbf{v}\| = 0$, so the desired equality holds. For the remainder of the proof, we will assume that $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors.

Consider the function $f(t) = \|t\mathbf{u} + \mathbf{v}\|^2$. Clearly, $f(t) \geq 0$ for all $t \in \mathbb{R}$. Further, $f(t) = 0$ for some real $t$ exactly when $\|t\mathbf{u} + \mathbf{v}\|^2 = 0$, which is to say, when $\mathbf{v} = -t\mathbf{u}$ for some real number $t$. That is, equality holds exactly when one vector is a scalar multiple of the other.

For all real $t$,

\[
\|t\mathbf{u} + \mathbf{v}\|^2 = (t\mathbf{u} + \mathbf{v}) \cdot (t\mathbf{u} + \mathbf{v}) = t^2\mathbf{u} \cdot \mathbf{u} + 2t\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = \|\mathbf{u}\|^2 t^2 + (2\mathbf{u} \cdot \mathbf{v})t + \|\mathbf{v}\|^2
\]

Let $a = \|\mathbf{u}\|^2 > 0$ since $\mathbf{u}$ is nonzero; let $b = 2\mathbf{u} \cdot \mathbf{v}$; and let $c = \|\mathbf{v}\|^2$. The condition $b^2 - 4ac \leq 0$ is equivalent to $(2\mathbf{u} \cdot \mathbf{v})^2 - 4\|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \leq 0$. Equivalently, $(\mathbf{u} \cdot \mathbf{v})^2 \leq \|\mathbf{u}\|^2 \|\mathbf{v}\|^2$. Taking the square root of each side produces the desired inequality. Equality occurs in each of these inequalities exactly when $\|t\mathbf{u} + \mathbf{v}\|^2 = 0$ for some real $t$, which is to say, when one vector is a scalar multiple of the other.