Guidelines for Writing Mathematical Proofs
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(Portions of this document were adapted from the following text: T. Sundstrom, Mathematical Reasoning: Writing and Proof, Prentice Hall, Inc., 2003.)

Here are some guidelines for writing a correct mathematical proof. Failure to follow these guidelines may result in a loss of points, even if the gist of your proof is correct.

1. Use correct English grammar and punctuation.

   I cannot emphasize enough how CRUCIAL it is for one to use complete, grammatically correct sentences and correct punctuation when writing proofs. In her book, Eats, Shoots and Leaves (NY: Gotham Books, 2004), Lynne Truss writes:

   We have a language that is full of ambiguities; we have a way of expressing ourselves that is often complex and allusive, poetic and modulated; all our thoughts can be rendered with absolute clarity if we bother to put the right dots and squiggles between the words in the right places. Proper punctuation is both the sign and the cause of clear thinking.

   The same is true of proper grammar. In the complex realm of mathematics it is absolutely essential to eliminate ambiguities.

   Write proofs using complete, non-run-on sentences and correct punctuation!

As part of this, make sure to:

(a) Make correct use of capital vs. lowercase letters.

   i. As is customary in writing in English, the first words in sentences should be capitalized.

   ii. Moreover, in math, the lowercase and uppercase versions of letters are NOT interchangeable when referring to named mathematical objects. For instance, one might use $F$ to denote a differentiable function, and $f$ to denote its derivative: generally, in this case, $F \neq f$. So if your variable is $r$, never denote it using the capital letter $R$; similarly, if your variable is $G$, do not denote it using the lowercase letter $g$.

Because of confusion that can arise when trying to simultaneously follow both of these rules, it is customary to avoid beginning sentences with names of mathematical objects.
(b) End sentences with appropriate punctuation marks.
(c) Take care to use words/phrases such as “hence,” “thus,” “therefore,” “if,” “only if,” “implies,” and “because” correctly.

For instance, if statements \( p \) and \( q \) are both true, but \( q \) would be true even if \( p \) weren’t true, then it is not appropriate to write “Since \( p, q \).” (For example, it would be unusual and almost certainly incorrect to include the following statement in a proof: “Since \( x \) is a prime number, 4 is divisible by 2.”)

A good trick to use when trying to decide whether or not your grammar is correct is to read what you have written aloud. Things that look fine on paper are often clearly recognizable as non-sentences when you try to read them out loud.

2. Make sure that what you are communicating is unambiguous.
For instance, the mathematical symbol \( \in \) means “is an element of” or “be an element of”. Notice that the phrase

\[ x \in S \]

is ambiguous when used on its own, and should not be used by itself. What are you trying to communicate with that phrase? Are you choosing an arbitrary element \( x \) in a set \( S \)? In this case, you could make this clear by using the sentence:

Let \( x \in S \).

Or are you, instead, noticing or concluding that the object \( x \) is an element of \( S \)? In this case, you could use the sentence

Notice that \( x \in S \).

or the sentence

We conclude that \( x \in S \).

3. Begin with a statement of your assumptions. You also might want to include, immediately following this, a carefully worded statement of the result to be proven. For example, suppose you are asked to prove the following theorem:

**Theorem:** If \( x \) is an odd integer, then \( x^2 \) is an odd integer.

You might begin your proof as follows:

**Proof.** Assume that \( x \) is an odd integer. We want to show that \( x^2 \) is an odd integer.

Alternatively, you could use the word “let”:

**Proof.** Let \( x \) be an odd integer. We want to show that \( x^2 \) is an odd integer.
4. **The word “let” is your friend.** You will frequently want to use the word “let” when setting up a context for what you are talking about. For instance, suppose you want variables $x$ and $y$ to represent positive integers which sum to 5. It is not enough to simply write down

$$x + y = 5.$$ 

If this is all you write, your reader doesn’t know if s/he’s reading a hypothesis or a conclusion; in addition, s/he doesn’t know what $x$ and $y$ are. Are they arbitrary complex numbers? Arbitrary real numbers? Specific rational numbers? Necessarily negative? If you don’t indicate what they are, your reader doesn’t know! But it’s easy enough to clarify things for your reader: simply write down something like

Let $x$ and $y$ be positive integers, with $x + y = 5$.

Then everything is clear.

5. **Use the royal “we”**. By convention, mathematicians tend to use the word “we” rather than “I” when writing proofs. For instance, you might write:

In this proof, we will show that the number $\pi$ is irrational.

The idea is that this gives the reader the feeling of having a partner when going through a proof. Unlike the first four guidelines, this guideline need not be followed in order for the proof you write to be correct; however, it is a convention which is nearly universally followed, and as it promotes the notion of camaraderie, it can be a pleasant guideline to follow!

6. **Keep your reader informed**. If you are going to use a specific kind of proof-technique, such as induction or proof by contradiction, it is helpful to let your reader know this. For example, you might simply tell your reader “We will use proof by contradiction”, after stating your original assumptions and the result you want to prove.

In addition, provide your reader with justification for every assertion made. For instance, you may refer to a hypothesis of a statement you are trying to prove; the definition of an object; or a previously proven theorem or result.

7. **Make things easy to read and follow**. In particular:

   - **Write at least one draft of your proof before writing your final draft.** This will help ensure that your final proof is easy to read, without unnecessary marks or erasures; it will also help you to write mathematically correct proofs, as you may find mathematical errors in your previous drafts.
• **Write clearly and neatly.** I cannot recommend strongly enough that you use pencil when writing your proofs: even in final drafts, it is easy to make typos. If you do make a mistake, erase your error thoroughly, so that no trace of it remains on the paper; **DO NOT CROSS WORK OUT.** If you feel you must use pen, make sure that you are writing a final draft of your proof: I should not see any white-out or scratch-outs on your homework.

• **Display important equations or mathematical expressions.** To make things easy to read, it is very often helpful to center significant equations, expressions, or algebraic manipulations on the page, leaving space above and below the centered material. For instance, suppose you want to find the value of $\lim_{h \to 0} \frac{s(3 + h) - s(3)}{h}$. Instead of awkwardly cramming your manipulations into one line of text, it is better to demonstrate your calculations as follows:

$$\lim_{h \to 0} \frac{s(3 + h) - s(3)}{h} = \lim_{h \to 0} \frac{(3 + h)^2 - 1 - 8}{h} = \lim_{h \to 0} \frac{9 + 6h + h^2 - 1 - 8}{h} = \lim_{h \to 0} \frac{6h + h^2}{h} = \lim_{h \to 0} (6 + h) = 6.$$

• **Let your reader know when your proof is starting.** The easiest way might be to simply skip a line, and then write **Proof.** at the beginning of your proof!

• **Let your reader know when your proof is finished.** To do this, you may use a symbol like $\blacksquare$, or the letters QED (which stand for the Latin phrase, *Quod Erat Demonstrandum*, which, in turn, translates into the phrase “Which was to be demonstrated”).

• **Do not begin a sentence with a mathematical symbol.** This is probably the least significant of the guidelines, but it is a convention that mathematicians tend to follow. In fact, in formal mathematics writing, it is also conventional to avoid the use of symbols ∀ (for every/all), ∃ (there exists), ∋ (such that), and ∴ (therefore). You may feel free to use these symbols when writing up your homework assignments for this class, but do **NOT** use a colon to mean “such that” outside of set-builder notation.