Math 455 Lecture Notes

0 Introduction

0.3 Basic Set Theory

0.3.1 Sets

- **Definition.** A set is an unordered collection of objects, called elements or members of the set. Exs, including roster and set-builder notation. What it means for a set to be well-defined. “Element of” notation: $\in$.

- $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$, $\mathbb{N}$ := \{1, 2, \ldots\}, $\mathbb{R}^+$, $\mathbb{R}^-$, $\mathbb{R}^*$, etc.

- $\emptyset$: universal sets $U$

- Subsets ($\subseteq$ or $\subset$) and proper subsets ($\varsubsetneq$). Note $\emptyset \subseteq S$ and $S \subseteq S$ for every set $S$; equal sets

- Power sets $\mathcal{P}(S)$. Ex: $\mathcal{P}(\emptyset) = \{\emptyset\}$, $\mathcal{P}\{1, 2\} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$.

- Venn diagrams, unions and intersections, disjoint sets, set difference $B - A = B \setminus A$ (aka, the relative complement of $A$ in $B$) the complement of $A$, $A^c (= U - A)$. Generalized unions and intersections. Mutually disjoint.

- Exs:
  1. Let $I = \{1, 2, 3, \ldots, 10\}$, and for each $n \in I$, let $A_n = (n - 1, n]$. Find $\bigcup_{n \in I} A_n$ and $\bigcap_{n \in I} A_n$. **Solutions.** They are, respectively, $(0, 10]$ and $\emptyset$.
  2. For each $r \in (0, 1)$, define set $E_r = [0, r)$. Find $\bigcup_{r \in (0, 1)} E_r$ and $\bigcap_{r \in (0, 1)} E_r$. **Solutions.** They are, respectively, $[0, 1)$ and $\{0\}$.
  3. For each $r \in (1, \infty)$, define set $X_r = (1/r, r)$. Find $\bigcup_{r \in (1, \infty)} X_r$ and $\bigcap_{r \in (1, \infty)} X_r$. **Solutions.** They are, respectively, $(0, \infty)$ and $\{1\}$.

- (On Slide 1 of 455slides.pdf) **Theorem.** Let $U$ be a universal set, and let $A$, $B$, $C$, and $D$ be subsets of $U$. Then
  
  $A \cap B = B \cap A$ (Commutative Law);
  $A \cup B = B \cup A$ (Commutative Law);
  $A \cap (B \cap C) = (A \cap B) \cap C$ (Associative Law);
  $A \cup (B \cup C) = (A \cup B) \cup C$ (Associative Law);
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive Law);
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributive Law);
  $C - (A \cup B) = (C - A) \cap (C - B)$ (De Morgan’s Law); and
  $C - (A \cap B) = (C - A) \cup (C - B)$ (De Morgan’s Law).
Proof of the first-listed Distributive Law: \(A \cap (B \cup C) = (A \cap B) \cup (A \cap C)\):

Let \(x \in A \cap (B \cup C)\). Then \(x \in B \cup C\), so either \(x \in B\) or \(x \in C\). If \(x \in B\), then \(x \in A \cap B\); if \(x \in C\), then \(x \in A \cap C\). So \(A \cap (B \cap C) \subseteq (A \cap B) \cup (A \cap C)\). Conversely, assume that \(x \in (A \cap B) \cup (A \cap C)\). Then either \(x \in A \cap B\), in which case \(x \in A \cap (B \cup C)\), or \(x \in A \cap C\), in which case \(x \in A \cap (B \cup C)\).

- These laws all generalize. In particular:

**Theorem.** Let \(\{E_\alpha\}_{\alpha \in A}\) be an indexed family of subsets of set \(X\). Then:

\[
E \cap \left(\bigcup_{\alpha \in A} E_\alpha\right) = \bigcup_{\alpha \in A} (E \cap E_\alpha) \quad \text{(Dist. Law)}
\]
\[
E \cup \left(\bigcap_{\alpha \in A} E_\alpha\right) = \bigcap_{\alpha \in A} (E \cup E_\alpha) \quad \text{(Dist. Law)}
\]
\[
X - \bigcup_{\alpha \in A} E_\alpha = \bigcap_{\alpha \in A} (X - E_\alpha) \quad \text{(De Morgan’s Law)}
\]
\[
X - \bigcap_{\alpha \in A} E_\alpha = \bigcup_{\alpha \in A} (X - E_\alpha) \quad \text{(De Morgan’s Law)}
\]

- Cartesian products, ordered pairs, generalized Cartesian products, ordered \(n\)-tuples. Notation: \(\mathbb{R}^n\).

### 0.3.2 Induction

- **Example.** Prove that \(1 + 2 + \cdots + n = \frac{n(n+1)}{2}\) for every \(n \in \mathbb{N}\).

**Proof:** Since \(1 = \frac{1(2)}{2}\), our base case holds. Next, let \(n\) be a positive integer and assume that \(1 + 2 + \cdots + n = \frac{n(n+1)}{2}\). Then

\[
1+2+\cdots+n+(n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}. \quad \square
\]

### 0.3.3 Functions

- Def. function/mapping (conventional notation—not the formal definition), domain \((\text{Dom } f)\), codomain, range \((\text{Range } f)\). Sometimes domains are implicitly defined.

- A function is **real-valued** if its codomain is \(\mathbb{R}\).

- Exs (include identity function \(i_A : A \to A\) and projections \(p_1 : A \times B \to A\) and \(p_2 : A \times B \to B\)).

- Given \(f : A \to B\), \(C \subseteq A\) and \(D \subseteq B\), def. of the *image* of \(C\) \((f(C))\) and the *preimage* or *inverse image* of \(D\) \((f^{-1}(D))\). **Note:** I recommend that you don’t use the book’s notation for the preimage, nor that you call it an inverse image.

- Exs.

- Def. of \(f : A \to B\) being **one-to-one** (aka *injective*), **onto** (aka *surjective*), and/or a **bijection** (bijective).
• How do you decide/prove whether or not a function is injective, surjective, or bijective.

• Def. func comp. Func comp. is assoc.

• Theorem. If \( f \) and \( g \) are surjective [resp., injective, resp., bijective], then so if \( g \circ f \).

   Proof. You prove on your homework that \( f, g \) surjective implies \( g \circ f \) is surjective. We here do the proof that \( f, g \) injective implies \( f \circ g \) injective.

0.3.5 Cardinality

• Definition. Set \( S \) has cardinality \( n \), and is said to be finite, if there exists a bijection from \( S \) to the set \( \{1, 2, \ldots, n\} \). The empty set is also considered finite, with cardinality 0. Sets that are not finite are infinite.

An infinite set \( S \) is said to be countably infinite if there exists a bijection from \( S \) to \( \mathbb{N} \); in this case we say its cardinality is \( \aleph_0 \). It’s countable if it’s finite or countable infinite; otherwise, it’s uncountably infinite, or simply uncountable.

• Exs: \( \mathbb{N} \) and \( \mathbb{Z}^- \) are countable. So are \( \mathbb{Z}, \mathbb{Z}^2, \ldots, \mathbb{Z}^n \ (n \in \mathbb{Z}^+) \). So is \( \mathbb{Q} \). Every subset of a countable set is countable.

Nonexs: \( \mathbb{R}, \mathbb{C} \), the interval \([0, 1]\), the set \( \{0, 1\}^\mathbb{N} \) are uncountable.

Theorem. Every infinite subset contains a countably infinite subset. (Proof omitted.)

1 Real Numbers

1.1 Basic properties

• Definition. An ordered set is a set \( S \), together with a relation \( < \) (called an order on \( S \)) such that

(i) For any \( x, y \in S \), exactly one of \( x < y \), \( x = y \), or \( y < x \) holds;

(ii) If \( x < y \) and \( y < z \), then \( x < z \) (that is, the relation is transitive)

We write \( x \leq y \) if \( x < y \) or \( x = y \). We define \( x > y \) if \( y < x \) and \( x \geq y \) if \( y < x \) or \( x = y \).

Note: For a formal definition of a relation, see Definition 0.3.4 in the book.

• Exs: \( \mathbb{N}, \mathbb{Z}, \mathbb{Q} \), and \( \mathbb{R} \) are ordered sets under the usual relation \( < \). \( \mathbb{C} \) is not an ordered set. You can order \( \mathbb{C} \) but not in a useful way.

• Definition. Let \( S \) be an ordered set and let \( E \subseteq \mathbb{R} \).
(i) We say $E$ is bounded above if there exists $M \in S$ such that $x \leq M$ for every $x \in E$. Such a $b$ is called an upper bound of $E$.

(ii) We say $E$ is bounded below if there exists $m \in S$ such that $x \geq m$ for every $x \in E$. Such an $m$ is called an lower bound of $E$.

(iii) If there exists an upper bound $M_0$ of $E$ such that every upper bound of $E$ is greater than or equal to $M_0$, then we call $M_0$ the least upper bound or supremum of $E$. We write $\sup E = M_0$.

(iv) If there exists a lower bound $m_0$ of $E$ such that every lower bound of $E$ is less than or equal to $m_0$, then we call $m_0$ the greatest lower bound or infimum of $E$. We write $\inf E = m_0$.

(v) A subset $E$ is bounded if it is bounded both above and below.

**Notes:**

- $\sup E$ and $\inf E$ do not need to live in $E$; they need to live in $S$.
- A subset $E$ or an ordered set $S$ may have infinitely many upper/lower bounds, but has at most one infimum and at most one supremum.

**Definition.** Let $E$ be a subset of an ordered set $S$. An element $y \in E$ is a maximum [resp., minimum] element of $E$ if $y \geq x$ [resp., $y \leq x$] for every $x \in E$.

Note: Maximum [resp., minimum] elements, when they exist, are unique.

**Lemma.** Let $E$ be a subset of an ordered set $S$. If $y \in E$ is a maximum [resp., minimum] element of $E$, then $y = \sup E$ [resp., $y = \inf E$].

**Proof.** Suppose $y \in E$ is a maximum of $E$. Then by definition, $y$ is an upper bound for $E$. Moreover, if $M$ is any upper bound for $E$, then by definition of an upper bound, since $y \in E$ we must have $y \leq M$. Thus, $y = \sup E$. The rest of the proof is similar. \(\square\)

**Exs using what we know about real numbers, provided without proof:**

1. Picture
2. Let $A = \left\{0, \frac{1}{2}, \frac{2}{3}, \ldots\right\} \subseteq \mathbb{R}$. $A$ is bounded below by 0 and above by 1. It’s also bounded below by $-1$, and $-\pi$, and $-0.00001$, and bounded above by $\ldots$ sup $A = 1/2$ and inf $A = 0$.
3. $\mathbb{Z}^+ \subseteq \mathbb{R}$ is bounded below by 1 (and lots of other numbers), but isn’t bounded above. inf $\mathbb{Z}^+ = 1$.
4. Let $B = \{r \in \mathbb{Q} : r > 0 \text{ and } r^2 < 2\} \subseteq \mathbb{R}$. inf $B = 0$ and sup $B = \sqrt{2}$.
5. How about open/closed/clopen intervals in $\mathbb{R}$? Do these have supers and/or infs?
6. Is $\emptyset \subseteq \mathbb{R}$ bounded? Does it have sup and/or inf in $\mathbb{R}$? Every real number is an upper and lower bound of $\emptyset$, but it has not sup or inf in $\mathbb{R}$.
• Definition. An ordered set $S$ has the least upper-bound property (LUBP) or completeness property if every nonempty subset $E$ of $S$ that is bounded above has a supremum in $S$.

• Exs/nonexs. $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}^+$ have the LUBP. $\mathbb{Q}$ doesn’t: e.g., consider the example $B$ above. Nor do $\mathbb{Z}^-$, $\mathbb{Q}^+$, $\mathbb{Q}^-$, $\mathbb{R}^-$, or $\mathbb{R}^*$. Why? Remember, the sup of $E$ has to live in $S$, not just in some universal set that contains $S$.

• The fact that $\mathbb{R}$ has the LUBP is one reason analysts work with it. The other is that it is a field.

• Definition. Let $S$ be a set. A binary operation on $S$ is a function from $S \times S$ to $S$.

• (On Slide 2 of 455slides.pdf) Definition. A field is a set $F$ equipped with two binary operations, $+$ and $\cdot$ (called addition and multiplication) such that the following axioms are satisfied:

1. $F$ is an abelian group under $+$. That is:
   
   $+$ is commutative and associative on $F$
   
   $F$ contains an element, denoted $0$, such that $x + 0 = x$ for all $x \in F$; $0$ is the additive identity element of $F$
   
   For every $x \in F$, there exists an element $-x$ of $F$ such that
   
   $x + (-x) = 0$; $-x$ is the additive inverse of $x$ in $F$.

2. Multiplication is commutative and associative on $F$.

3. $F$ contains a nonzero element, denoted $1$, such that $1x = x$ for all $x \in F$; $1$ is the multiplicative identity element or unity of $F$.

4. For every $0 \neq x \in F$, there exists an element $x^{-1}$ of $F$ such that
   
   $xx^{-1} = 1$; $x^{-1}$ is the multiplicative inverse of $x$ in $F$.

5. For all $x, y, z \in F$, $x(y + z) = xy + xz$ (multiplication in $F$ distributes over addition).

If $x$ and $y$ are elements of a field $F$, we may denote $x + (-y)$ more simply by $x - y$ and $xy^{-1}$ by $x/y$ (provided, of course, that $y \neq 0$, so $y^{-1}$ exists).

• Theorem. Let $F$ be a field and let $x, y \in F$. Then:

1. $-(-x) = x$ and $(x^{-1})^{-1} = x$;
2. $0x = 0$;
3. $(-x)y = -xy = x(-y)$;
4. $(-x)(-y) = xy$. 
\textbf{Proof.} The first two equalities are straightforward. Next, \(0x = (1-1)x = 1x - 1x = 0\). Then \(xy + (-x)y = (x - x)y = 0y = 0\), so \((-x)y = -xy\). Similarly, \(x(-y) = -xy\). Finally, \[-xy + (-x)(-y) = (-x)y + (-x)(-y) = -x(y - y) = -x0 = 0,\] so \((-x)(-y) = -(xy) = xy\). 

- Exs/nonexs. \(\mathbb{R}, \mathbb{Q}, \mathbb{C}\) are fields. \(\mathbb{Z}, \mathbb{N}, \mathbb{R}^{+}, \mathbb{M}_{n}(\mathbb{R}), GL(n, \mathbb{R})\) are not. When \(p\) is prime, \(\mathbb{Z}_p\) is a field under addition and multiplication mod \(p\), but \(\mathbb{Z}_n\) isn’t a field if \(n\) isn’t prime.

- \textbf{Definition.} A field \(F\) is an \textit{ordered field} if \(F\) is also an ordered set such that
  \begin{enumerate}
  \item For all \(x, y, z \in F\), \(x < y\) implies \(x + z < y + z\)
  \item For all \(x, y \in F\), \(x > 0\) and \(y > 0\) implies \(xy > 0\).
  \end{enumerate}
In other words, a field is an ordered field if its ordering is respected by its addition and multiplication.

- \textbf{Note:} Calling a set \(F\) an ordered field implicitly means you have defined three things on it: binary operations of addition and multiplication under which \(F\) is a field, and an order on \(F\) that respects these operations.

- Exs: \(\mathbb{R}\) and \(\mathbb{Q}\) are ordered fields. \(\mathbb{C}\) are \(\mathbb{Z}_p\) (\(p\) prime) fields but not ordered fields.

- \textbf{Definition.} Let \(F\) be an ordered field. Then we say \(x \in F\) is \textit{positive} if \(x > 0\), \textit{negative} if \(x < 0\), \textit{nonnegative} if \(x \geq 0\), and \textit{nonpositive} if \(x \leq 0\).

- \textbf{(On Slide 3 of 455slides.pdf) Ordered Field Theorem.} Let \(F\) be an ordered field and \(x, y, z \in F\). Then:
  \begin{enumerate}
  \item If \(x > 0\), then \(-x < 0\) (and vice versa);
  \item If \(x > 0\) and \(y < z\), then \(xy < xz\);
  \item If \(x < 0\) and \(y < z\), then \(xy > xz\);
  \item If \(x \neq 0\), then \(x^2 > 0\); in particular, then, \(1 > 0\), and \(x^2 \geq 0\) for all \(x \in F\);
  \item If \(0 < x < y\), then \(0 < 1/y < 1/x\);
  \item If \(x \leq y\) and \(z \leq w\), then \(x + z \leq y + w\);
  \item If \(xy > 0\), then either both \(x\) and \(y\) are positive, or \(x\) and \(y\) are both negative.
  \end{enumerate}
\textbf{Proof.} We just prove (ii) and (vii), as examples. For (ii): Assume that \(x > 0\) and \(y < z\). By the definition of an ordered field, \(y < z\) implies that
\[ z - y > y - y = 0; \text{ then } x > 0 \text{ and } z - y > 0 \text{ imply } x(z - y) > 0, \text{ which means that } xz - xy > 0, \text{ or } xz > xy. \]

For (vii): If \( xy > 0 \), then neither \( x \) nor \( y \) can be zero (since in that case we’d have \( xy = 0 \)). If \( x > 0 \) and \( y < 0 \), then multiplying both sides of the second inequality by \( x \) yields, \( xy < x0 = 0 \), which is a contradiction. Similarly, if \( x < 0 \) and \( y > 0 \), we obtain a contradiction. Thus, either \( x \) and \( y \) are both positive or \( x \) and \( y \) are both negative.

\[ \begin{align*}
\textbf{• Definition and Theorem.} & \quad \text{Let } F \text{ be an ordered field with the LUBP} \\
& \quad \text{(that is, } F \text{ is an ordered field such that every nonempty subset } E \text{ of } F \text{ that is bounded above has a supremum in } F). \text{ Then } F \text{ also has the greatest lower bound property (GLBP): that is, given any nonempty subset } E \text{ of } F \text{ that is bounded below, } \inf E \text{ exists in } F. \\
\textbf{Proof.} & \quad \text{Suppose } \emptyset \neq E \subseteq F \text{ is bounded below. Then the set } -E := \{-x : x \in E\} \text{ is bounded above, so } \sup(-E) \text{ exists in } F. \text{ Let } M = \sup(-E). \text{ Then } -x \leq M \text{ for every } x \in E, \text{ so } x \geq -M \text{ for every } x \in E; \text{ thus, } -M \text{ is a lower bound for } E. \text{ Moreover, if } m \text{ is any lower bound for } E, m \leq x \text{ for each } x \in E \text{ implies } -m \geq -x \text{ for each } x \in E, \text{ implying that } -m \text{ is an upper bound for } -E, \text{ and hence is greater than or equal to } M; \text{ but then } -m \geq M \text{ implies } m \leq -M. \text{ So } -M \text{ is a greatest lower bound for } E. \quad \Box
\end{align*} \]

\[ \begin{align*}
\textbf{• Definition.} & \quad \text{Let } E \text{ be a subset of an ordered set } S. \text{ An element } y \in E \text{ is a maximum [resp., minimum] element of } E \text{ if } y \geq x \text{ [resp., } y \leq x\] for every } x \in E. \\
\textbf{Note:} & \quad \text{Maximum [resp., minimum] elements, when they exist, are unique.}
\end{align*} \]

\[ \begin{align*}
\textbf{• Lemma.} & \quad \text{Let } E \text{ be a subset of an ordered set } S. \text{ If } y \in E \text{ is a maximum [resp., minimum] element of } E, \text{ then } y = \sup E \text{ [resp., } y = \inf E\] . \\
\textbf{Proof.} & \quad \text{Suppose } y \in E \text{ is a maximum of } E. \text{ Then by definition, } y \text{ is an upper bound for } E. \text{ Moreover, if } M \text{ is any upper bound for } E, \text{ then by definition of an upper bound, since } y \in E \text{ we must have } y \leq M. \text{ Thus, } y = \sup E. \text{ The rest of the proof is similar. } \quad \Box
\end{align*} \]

1.2 The set of real numbers

\[ \begin{align*}
\textbf{• Theorem.} & \quad \text{Up to isomorphism, there exists a unique ordered field that has the LUBP and contains } \mathbb{Q}; \text{ we call that field } \mathbb{R}. \text{ \textbf{Proof.} Omitted.} \\
& \quad \text{Basically, you’re taking } \mathbb{Q} \text{ and adding all infs and sups of subsets of } \mathbb{Q}. \\
\textbf{• Notice that since } \mathbb{R} \text{ is an ordered field, } 0 < 1; \text{ thus, using the first part of the definition of an ordered field,} \\
& \quad 1 = 0 + 1 < 1 + 1 = 2;
\end{align*} \]

using this and induction yields

\[ 0 < 1 < 2 < 3 < \ldots \]
Similarly, using induction and part (v) of the Ordered Field Proposition,

\[ 1 > 1/2 > 1/3 > \cdots \]

- **Proposition.** Let \( x \in \mathbb{R} \) such that \( x \leq \varepsilon \) for all \( \varepsilon \in \mathbb{R} \) with \( \varepsilon > 0 \). Then \( x \leq 0 \).

**Proof.** Suppose \( x > 0 \). Let \( \varepsilon = x/2 \). Since \( x > 0 \) and \( 1/2 < 1 \), part (ii) of the Ordered Field Proposition yields that

\[ \varepsilon = x/2 = x(1/2) < x(1) = x, \]

which contradicts our assumption. So \( x \leq 0 \).

- **The Archimedean Property.** Let \( x, y \in \mathbb{R} \) with \( x > 0 \). Then there exists \( n \in \mathbb{N} \) such that \( nx > y \).

**Proof.** Note that proving this proves that there exists \( n \in \mathbb{N} \) such that \( n > y/x \). Suppose that \( \mathbb{N} \) is bounded above. Then since \( \mathbb{R} \) has the LUBP, \( b := \sup \mathbb{N} \) exists. Since \( b - 1 < b, b - 1 \) isn’t an upper bound for \( \mathbb{N} \). So there is a natural number \( m \) that is greater than \( b - 1 \). Then \( m + 1 \) is a natural number with \( m + 1 > (b - 1) + 1 = b \), which is a contradiction. Thus, \( \mathbb{N} \) isn’t bounded above. Thus, there is a natural number \( n \) that is larger than \( y/x \), as desired. \( \square \)

- **Corollaries.**

1. Let \( y \in \mathbb{R} \). Then there exists \( n \in \mathbb{N} \) such that \( n > y \). (Proof: Use \( x = 1 \).)

2. Let \( 0 < x \in \mathbb{R} \). Then there exists \( n \in \mathbb{N} \) such that \( 1/n < x \). (Proof: Use \( y = 1 \).)

- **Definition. Theorem.** (\( \mathbb{Q} \) is dense in \( \mathbb{R} \)) Let \( x, y \in \mathbb{R} \) with \( x < y \). Then there exists \( r \in \mathbb{Q} \) such that \( x < r < y \).

Assume \( x \geq 0 \). Since \( x < y \) implies \( y - x > 0 \), by the Archimedean Property there exists \( n \in \mathbb{N} \) such that \( 1/n < y - x \), that is, such that \( x + 1/n < y \). Consider the set \( A := \{ k \in \mathbb{N} : k/n > x \} \). Since \( A \) is a nonempty subset of \( \mathbb{N} \), it has a least element, \( m \). Since \( m \in A \), \( m/n > x \); moreover, since \( m \) is the least element of \( A \), \( m - 1 \notin A \), so that \( (m - 1)/n \leq x \), implying that

\[ m/n = (m - 1)/n + 1/n \leq x + 1/n < y. \]

So \( r = m/n \in \mathbb{Q} \) with \( x < m/n < y \).

On the other hand, assume that \( x < 0 \). If \( y > 0 \), then \( r = 0 \) works. If \( y \leq 0 \), then we have \( 0 \leq -y \leq -x \), so by what we’ve already proven, there exists \( q \in \mathbb{Q} \) with \( -y < q < -x \); letting \( r = -q \), then, we’re done. \( \square \)

- We return to discussing sups and infs, now of subsets of \( \mathbb{R} \).
• **Theorem and Definition.** For every $x \geq 0$ and $n \in \mathbb{N}$, there exists a unique positive real number $r$ such that $r^n = x$. We denote this number by $x^{1/n}$ or $\sqrt[n]{x}$.

**Proof.** The gist is that $x^{1/n} := \sup\{r \in \mathbb{R} : r^n = x\}$. □

• **Theorem.** Let $n \in \mathbb{N}$ and $x, y > 0$. Then $\sqrt[n]{x} \sqrt[n]{y} = \sqrt[n]{xy}$.

**Proof.** $\sqrt[n]{x} \sqrt[n]{y} > 0$ and $(\sqrt[n]{x} \sqrt[n]{y})^2 = xy$. □

• **Theorem.** Let $\emptyset \neq A \subseteq \mathbb{R}$ be bounded above. Then for every $\varepsilon > 0$, there exists an $x \in A$ such that

$$(\sup A) - \varepsilon < x < \sup A.$$  

**Proof.** Let $s := \sup A$, and consider the set $B := A \cap (s - \varepsilon/2, s)$. Since every element of $A$ is less than or equal to $s$, if $B$ is empty, then every element of $A$ is less than or equal to $s - \varepsilon/2$, which contradicts the fact that $s = \sup A$. So there exists $x \in B$. Then $x \in A$ with $s - \varepsilon < s - \varepsilon/2 < x < s$. □

**Note:** Another way of stating this is that if $\emptyset \neq A \subseteq \mathbb{R}$ is bounded above and $y < \sup A$, then there exists $x \in A$ with $y < x < \sup A$.

• **Definition.** Let $x \in \mathbb{R}$ and $A \subseteq \mathbb{R}$. Then we define $x + A := \{x + a : a \in \mathbb{R}\}$ and $xA := \{xa : a \in \mathbb{R}\}$.
• **Theorem.** Let \( \emptyset \neq A \subseteq \mathbb{R} \). If \( x \in \mathbb{R} \) and \( A \) is bounded above [resp., below] then \( \text{sup}(x + A) = x + \text{sup} A \) [resp., \( \text{inf}(x + A) = x + \text{inf} A \)].

**Proof.** Suppose \( b \) is an upper bound for \( A \). That is, \( y \leq b \) for all \( y \in A \). Then \( x + y \leq x + b \) for all \( y \in A \), so \( x + b \) is an upper bound for \( x + A \). In particular, \( x + \text{sup} A \) is an upper bound for \( x + A \), so \( \text{sup}(x + A) \leq x + \text{sup} A \).

On the other hand, if \( c \) is an upper bound for \( x + A \), then \( x + y \leq c \) for all \( y \in A \); in particular, \( x + y \leq \text{sup}(x + A) \) for all \( y \in A \), so that for all \( y \in A \), \( y \leq \text{sup}(x + A) - x \); thus, \( \text{sup}(x + A) - x \) is an upper bound for \( A \), and thus \( \text{sup} A \leq \text{sup}(x + A) - x \), or \( \text{sup}(x + A) \geq \text{sup} A + x \).

Hence, \( \text{sup}(x + A) = x + \text{sup} A \).

The proof for the infima is similar. \( \Box \)