Algebraic Properties of $\mathbb{R}^n$ and $\mathbb{M}_{m \times n}(\mathbb{R})$
(Sections 1.3 & 2.1)
Math 331

Let $u, v, w \in \mathbb{R}^n$, let $c, d \in \mathbb{R}$, and let $0$ denote the zero vector in $\mathbb{R}^n$. Then the following hold:

(i) $u + v = v + u$
(ii) $(u + v) + w = u + (v + w)$
(iii) $u + 0 = 0 + u = u$
(iv) $u + (-u) = -u + u = 0$
(v) $c(u + v) = cu + cv$
(vi) $(c + d)u = cu + du$
(vii) $c(du) = (cd)u$
(viii) $1u = u$

Let $A, B, C \in \mathbb{M}_{m \times n}(\mathbb{R})$, let $c, d \in \mathbb{R}$, and let $0$ denote the $m \times n$ zero matrix. Then the following hold:

(i) $A + B = B + A$
(ii) $(A + B) + C = A + (B + C)$
(iii) $A + 0 = 0 + A = A$
(iv) $A + (-A) = -A + A = 0$
(v) $c(A + B) = cA + cB$
(vi) $(c + d)A = cA + dA$
(vii) $c(dA) = (cd)A$
(viii) $1A = A$

Let $A$ be an $m \times n$ matrix, let $B$ and $C$ be matrices of the appropriate sizes, and let $c \in \mathbb{R}$. Then the following hold:

(i) $A(BC) = (AB)C$
(ii) $(A + B)C = AC + BC$
(iii) $A(B + C) = AB + AC$
(iv) $c(AB) = (cA)B$
(v) $I_mA = AI_n = A$

WARNINGS:

- The matrix product $AB$ does not exist if the number of columns of $A$ doesn’t equal the number of rows of $B$!
- In general, $AB \neq BA$ (in fact, $AB$ may exist while $BA$ doesn’t, or vice versa!) If $AB = BA$, we say $A$ and $B$ commute.
- The Zero Product Principle does not hold for matrices: that is, $AB = 0$ does NOT imply $A = 0$ or $B = 0$! Ex: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$.
- In general, left and right cancellation properties don’t hold for matrices: that is, in general, $AB = AC$ does NOT imply $B = C$, and $BA = CA$ does NOT imply $B = C$. 