UNIVERSAL REFLECTION SUBGROUPS AND EXPONENTIAL GROWTH IN COXETER GROUPS

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ABSTRACT. We investigate the imaginary cone in hyperbolic Coxeter systems in order to show that any Coxeter system contains universal reflection subgroups of arbitrarily large rank. Furthermore, in the hyperbolic case, the positive spans of the simple roots of the universal reflection subgroups are shown to approximate the imaginary cone (using an appropriate topology on the set of roots), answering a question due to Dyer [9] in the special case of hyperbolic Coxeter systems. Finally, we discuss growth in Coxeter systems and utilize the previous results to extend the results of [16] regarding exponential growth in parabolic quotients in Coxeter groups.

1. Introduction and Preliminaries

By a universal Coxeter system, we mean one with no braid relations (the underlying group is a free product of cyclic groups of order 2). The main result of this article is that all irreducible, finite rank, infinite, non-affine Coxeter systems have universal reflection subgroups of arbitrarily large rank. To prove this, we use the imaginary cone, which was introduced by Dyer ([8]) in order to prove a conjectured characterization of coverings in the dominance order on the root system. Dyer

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raised the question of whether the imaginary cone of an infinite, irreducible, non-affine Coxeter system can be approximated by imaginary cones of universal reflection subgroups. We show this is the case for hyperbolic Coxeter groups, and then our main result follows from the fact that all finite rank, infinite, non-affine Coxeter systems contain a parabolic subsystem that is hyperbolic ([11]).

The paper is organized as follows. For the rest of this section, we introduce the necessary tools and terminology with respect to Coxeter groups, root systems, and reflection subgroups. In section 2, we introduce some some cones in Coxeter systems as well as a topology on cones. We also provide some auxiliary results about the imaginary cone of a Coxeter system; all of these results are developed and proven in [9]. We then utilize these results to prove Proposition 2.7.1 and Theorem 2.7.2, our main results describing the existence of large-rank universal reflection subgroups of infinte, non-affine Coxeter groups and establish a connection between the imaginary cones of these subgroups with the cone of the group. Finally, in section 3 we investigate growth types of subsets in Coxeter groups and use the large reflection subgroups from section 2 to answer a question posed in [16] by proving Theorem 3.4.1, which essentially states that parabolic quotients of all finite rank, irreducible, infinite, non-affine Coxeter systems have exponential growth.

We begin with the necessary preliminaries with respect to Coxeter systems.

1.1. **Coxeter Systems, Root Systems and Reflection Subgroups.**

Let (W, S) be a finite rank Coxeter system where $m(s_i, s_j)$ represents the order of $s_i s_j$ for all i, j. Without loss of generality, we will assume that (W, S) is realized in its standard reflection representation on a real vector space V with Π denoting the set of simple roots and $(-,-) : V \times V \to \mathbb{R}$ representing the associated bilinear form given by $(\alpha, \beta) = -\cos \frac{\pi}{m(s_\alpha, s_\beta)}$ if $m(s_\alpha, s_\beta) \neq \infty$ and $(\alpha, \beta) \leq -1$ otherwise. Then we denote the roots and positive roots by Φ and Φ^+ respectively. Let $A := A_{(W,S)}$ be the matrix of the bilinear form $(-,-)$. That is, A is a symmetric matrix with $A_{i,j} := A_{s_i,s_j} = (\alpha_i, \alpha_j)$ where α_k is the simple root associated to s_k. Let $T = \cup_{w \in W} wSw^{-1}$ be the set of reflections of W.

We call a subgroup W' of W a *reflection subgroup* of W if it is generated by the reflections it contains, $W' = \langle W' \cap T \rangle$. It was shown by Dyer ([10]) and Deodhar ([7]) independently that any reflection subgroup is also a Coxeter system; moreover, any reflection subgroup has a canonical set of Coxeter generators $\chi(W') = \{t \in T | \; N(t) \cap W' = \{t\}\}$, where $N : W \to \mathscr{P}(T)$ is the reflection cocycle defined in [10]. Since any reflection subgroup is also a Coxeter system $(W', \chi(W'))$, we let $\Phi_{W'}$, $\Phi^+_{W'}$, $\Pi_{W'} \subseteq \Phi$ be the set of roots, positive roots, and simple roots for $(W', \chi(W'))$ sitting inside the root system for (W, S). We have that $\Phi_{W'} = \{\alpha \in \Phi | \; s_\alpha \in W'\}$. For any reflection subgroup, $W' \leq W$, we get a corresponding length function $l_{(W', \chi(W'))} : W' \to \mathbb{N}$; when $W' = W$, we have the standard length function $l := l_{(W,S)}$.

We say (W, S) is a *hyperbolic* Coxeter system if the bilinear form $(-,-)$ on V has inertia $(n-1, 1, 0)$ and every proper standard parabolic Coxeter system is of finite or affine type (see [12]).

We say that $\alpha \in \Phi$ *dominates* $\beta \in \Phi$ written $\beta \preceq \alpha$ if, for all $w \in W$, $w(\alpha) \in \Phi^-$ implies that $w(\beta) \in \Phi^-$. Dominance order is a partial order on Φ and we will utilize a few key properties that can be found in [2], [3], and [8]. For any interest in standard results in Coxeter groups, consult [2] or [12].

2. Large Universal Reflection Subgroups

2.1. **Convex Sets and the Hausdorff Metric.** In this work, we will define a few different cones, and so we need some terminology regarding convexity and polyhedral cones. For a reference on such terms, consult [1]. In particular, we describe our terminology involved with convex sets. Let V be a real vector space. For any set $M \subseteq V$ we call the set of all convex combinations of elements of M the convex hull of M and we denote it by $\mathrm{conv}(M)$.

For any two elements $x, y \in V$, we denote

$$[x, y] := \{\lambda x + (1-\lambda)y \mid 0 \le \lambda \le 1\}.$$

We say a set $C \subset V$ is *convex* if, for all $x, y \in C$ $(x \neq y)$, $[x, y] \subset C$.

Assume that V is finite dimensional, so that $V \cong \mathbb{R}^n$. Then V has the usual distance function as a metric $d : V \times V \to \mathbb{R}$. Let $Y \subseteq V$ and $x \in V$. Define

$$d(x, Y) = \inf_{y \in Y} d(x, y) \tag{2.1}$$

Then for any $X \subseteq V$ and $Y \subseteq V$, we define

$$d(X,Y) = \sup_{x\in X} d(x,Y) \tag{2.2}$$

Definition 2.1.1. Let $K := \{X \subseteq V \mid X \text{ is compact}\}$.

The following proposition can be found in [14, §21 VII].

Proposition 2.1.2. *Consider the distance function defined in* (2.2). *We create a new distance function on K given by*

$$d_K(X,Y) = \max\{d(X,Y), d(Y,X)\}.$$

Then d_K defines a metric on K.

The metric defined above is known as the Hausdorff metric (or Hausdorff distance).

2.2. **Approximation of a Sphere by Polytopes.** Now, suppose that S^d is the d-sphere sitting inside of $\mathbb{R}^{d+1}$. We want to show that S^d can be approximated by the boundary of a convex hull of a finite number of points. For any set $A \subseteq \mathbb{R}^n$, let ∂A denote the boundary of A. To make the previous statement more concrete, we have the following lemma; we include a proof here because we could not find a suitable reference.

Lemma 2.2.1. *There exists a sequence of compact, convex sets P_m with $P_m = conv(\{x_1, ..., x_m\})$ where $\{x_1, ..., x_m\} \subset S^d$ and $m < \infty$ such that $d_K(\partial P_m, S^d) \to 0$ as $m \to \infty$, i.e. $\partial P_m \to S^d$ in the metric space K.*

Proof. Let $\epsilon > 0$ be given. Now, for any point $z \in \mathbb{R}^{d+1}$, define $\mathscr{B}_a(z)$ to be the open ball of radius a centered at z. Then we define

$$\mathcal{U} := \{\mathscr{B}_{\frac{\epsilon}{2}}(z) |\ z \in S^d\}.$$

Clearly $S^d \subset \bigcup_{U \in \mathcal{U}} U$. Since S^d is compact, we can find a finite open subcover of $\mathcal{U}$ call it $\mathcal{U}' := \{U_1, ..., U_m\}$ (where $m = m_\epsilon$ is dependent on ϵ).

Now, each U_i is a ball of radius $\epsilon/2$ centered at $x_i \in S^d$. Thus, we have a finite set of points $\{x_1, ..., x_m\} \subset S^d$. Let $P_m := \text{conv}(\{x_1, ..., x_m\})$. Let ∂P_m denote the boundary of P_m.

Suppose $y \in \partial P_m$. Now, there exists a minimal set $I := \{x_{i_1}, ..., x_{i_k}\} \subset \{x_1, ..., x_m\}$ such that $y \in \text{conv}(\{x_{i_1}, ..., x_{i_k}\}) = \text{conv}(I) \subset \partial P_m$. Then, we can choose a supporting affine hyperplane H of P_m such that $I \subset P_m \cap H$. Let H^+ and H^- represent the positive and negative half-spaces associated to H. Suppose, without loss of generality, that $P_m \subset H^-$. Let $H' := \text{int}(H^+)$ and let u be an outer unit normal vector of H (u points into H^+). By assumption, $P_m \cap H' = \emptyset$ and so $\partial P_m \cap H' = \emptyset$ and $\{x_1, ..., x_m\} \cap H' = \emptyset$.

Next, for the sake of contradiction, we suppose that $d(y, S^d) \geq \epsilon$. Then let $L = H + \frac{\epsilon}{2}u$ be the affine hyperplane parallel to H with $d(H, L) = \frac{\epsilon}{2}$ and $L \subset H'$. It follows that $S^d \cap L^+ \neq \emptyset$, where L^+ is the positive half-space associated to L. Indeed, if $S^d \cap L^+ = \emptyset$, then $S^d \subseteq L^-$ and so $d(P_m \cap H, S^d) \leq d(H, L)$. This would imply that

$$d(y, S^d) \leq \sup_{x \in P_m \cap H} d(x, S^d) = d(P_m \cap H, S^d) \leq d(H, L) = \frac{\epsilon}{2} < \epsilon,$$

which is not true by assumption. Thus, we let $z \in S^d \cap L^+$. Then $\mathscr{B}_{\frac{\epsilon}{2}}(z) \subset H'$ by construction. However, $z \in S^d$ so $z \in U_i$ for some $U_i \in \mathcal{U}'$ so $x_i \in \mathscr{B}_{\frac{\epsilon}{2}}(z) \subset H'$ which is a contradiction. Therefore, $d(y, S^d) < \epsilon$.

Since y is arbitrary, we can conclude that $d(\partial P_m, S^d) < \epsilon$. Now, suppose $y \in S^d$, then $y \in U_i$ for some i. Thus,

$$d(y, \partial P_m) = \inf_{z \in \partial P_m} d(y, z) \leq d(y, x_i) < \frac{\epsilon}{2}.$$

Again, since y is arbitrary, we can conclude that $d(S^d, \partial P_m) < \frac{\epsilon}{2}$. Thus we have determined that $d_K(\partial P_m, S^d) = \max\{d(S^d, \partial P_m), d(\partial P_m, S^d)\} < \max\{\frac{\epsilon}{2}, \epsilon\} = \epsilon$ as desired. □

2.3. **Cones in Coxeter Systems.** For the remainder of this section, let (W, S) be a Coxeter system with S finite. Therefore, according to section 1.1, we have V, $(-,-)$, Φ, Φ^+, and Π all defined.

Recall, for any reflection subgroup W', we let $\chi(W')$ be the set of canonical generators of W' as a Coxeter system. Then, according to section 1.1, we have a corresponding set of roots, positive roots, and simple roots for the Coxeter system $(W', \chi(W'))$ given by

$$\Phi_{W'} = \{\alpha \in \Phi |\, s_\alpha \in W'\}, \quad \Phi^+_{W'} := \Phi_{W'} \cap \Phi^+, \quad \Pi_{W'} = \{\alpha \in \Phi^+ |\, s_\alpha \in \chi(W')\}$$

respectively.

We now introduce several cones that appear inside of the vector space V introduced in section 1.1. For any W' a reflection subgroup of W, we define the fundamental chamber for W' on the vector space V by

$$\mathscr{C}_{W'} = \{v \in V \mid (v, \alpha) \geq 0 \text{ for all } \alpha \in \Pi_{W'}\}.$$

Then we denote the Tits cone of W' by $\mathscr{X}_{W'}$ where this is defined by

$$\mathscr{X}_{W'} = \bigcup_{w \in W'} w(\mathscr{C}_{W'}).$$

In addition to these we define the following:

$$\mathscr{K}_{W'} := \mathbb{R}_{\geq 0}\Pi_{W'} \cap -\mathscr{C}_{W'}, \qquad \mathscr{Z}_{W'} := \bigcup_{w \in W'} w(\mathscr{K}_{W'})$$

where $\mathscr{Z}_{W'}$ is known as the imaginary cone of W' on V (we only define it if S is finite). The definition of the imaginary cone here is taken from [9], which models it loosely on one characterization of imaginary cones of Kac-Moody Lie algebras ([13]). The results we use are analogs for general W of results proved by Kac for imaginary cones of Kac-Moody Lie algebras (for which, however, the proofs use imaginary roots, which have no counterpart here). We will refer to $\mathscr{C}_W, \mathscr{X}_W, \mathscr{K}_W$ and $\mathscr{Z}_W$ simply by $\mathscr{C}, \mathscr{X}, \mathscr{K}$ and $\mathscr{Z}$.

2.4. **Topology on Rays in Cones.** By a *ray* through v in V we mean a set $\{\lambda v \mid \lambda \in \mathbb{R}_{\geq 0}\}$. We let $\mathscr{R}$ denote the set of rays of V. We can give $\mathscr{R}$ a topology and a metric in the following way. Let B be compact convex body containing 0 in V, and let ∂B denote the boundary of B. Define the map $\varphi : \mathscr{R} \to \partial B$ by $\varphi(x) = x \cap \partial B$. It is clear that φ is a bijection. We thus declare φ to be a homeomorphism between $\mathscr{R}$ and ∂B, and this gives $\mathscr{R}$ a topology and a metric corresponding to ∂B in

V, with the topology independent of B. We also introduce the following subsets of $\mathscr{R}$. Let $\mathscr{R}_+ := \{\mathbb{R}_{\geq 0}\alpha \mid \alpha \in \Phi^+\}$ and let $\mathscr{R}_0 := \overline{\mathscr{R}_+} \setminus \mathscr{R}_+$. We say a ray, $\mathbb{R}_{\geq 0}\alpha$, is positive if $(\alpha, \alpha) > 0$ and isotropic if $(\alpha, \alpha) = 0$. We recall some facts about these sets (both proven in [9]).

Proposition 2.4.1. (1) *$\mathscr{R}_+$ consists of positive rays and is discrete in the subspace topology.*

(2) *$\mathscr{R}_0$ consists of isotropic rays and is closed in $\mathscr{R}$.*

(3) *$\mathscr{R}_0$ is the set of limit rays of $\mathscr{R}_+$.*

In addition, we have the following theorem.

Theorem 2.4.2. *The cone $\overline{\mathscr{Z}}$ is the convex closure of $\bigcup_{r \in \mathscr{R}_0} r \cup \{0\}$.*

This theorem has the following immediate corollary.

Corollary 2.4.3. *If W' is a finitely generated reflection subgroup of W, then $\overline{\mathscr{Z}_{W'}} \subseteq \overline{\mathscr{Z}_W}$.*

Remark. A much stronger fact, proven in [9], which we will not use, is that in the situation of the corollary above we actually have $\mathscr{Z}_{W'} \subseteq \mathscr{Z}_W$.

2.5. **Cones in Hyperbolic Coxeter Systems.** Suppose we fix a hyperbolic Coxeter system (W, S). We know that V has a basis $x_0, ..., x_n$ such that $(x_i, x_j) = 0$ for $i \neq j$, $(x_i, x_i) = 1$ for all $i \in \{1, ..., n\}$, and $(x_0, x_0) = -1$. Then, according to [9], we have (after replacing x_0 by $-x_0$ if necessary) that

$$\overline{\mathscr{Z}} = \overline{\mathscr{L}}$$

where $\overline{\mathscr{L}} = \{\sum_{i=0}^{n} \lambda_i x_i \mid \lambda_0 \geq \sqrt{\sum_{i=1}^{n} \lambda_i^2}\}$.

Now, consider the setup as in section 2.4. Let $\mathscr{R}_\Pi \subset \mathscr{R}$ be the set of rays spanned by non-zero elements of $\mathbb{R}_{\geq 0}\Pi$. Since (W, S) is hyperbolic, the bilinear form is non-degenerate, and thus the interior of $\mathscr{C}$ is non-empty (see [9]). Take $\rho \in \text{int}(\mathscr{C})$ so that $(\rho, \alpha) > 0$ for all $\alpha \in \Pi$. Therefore, we may define a map

$$\tau : \mathbb{R}_{\geq 0}\Pi \setminus \{0\} \to H := \{v \in V | \ (v, \rho) = 1\}$$

given by $\tau(v) = v/(\rho, v)$. Then the image of τ is the set $P := \{v \in V | \ (\rho, v) = 1\}$, which is a convex polytope in the affine hyperplane H with points $(\rho, \alpha)^{-1}\alpha$ for $\alpha \in \Pi$ as vertices. From τ, we introduce a map on $\mathscr{R}$ as follows. For $r \in \mathscr{R}$, we let $\hat{\tau}(r) = \tau(x)$ for all $x \in r \setminus \{0\}$ (note $\hat{\tau}$ is independent of choice of $x \in r \setminus \{0\}$). With terminology from section 2.4 we may take B so that $P \subset \partial B$.

Now, we can see that $\overline{\mathscr{Z}} \cap H$ is a disk. Also, by Theorem 2.4.2 we see that $\overline{\mathscr{Z}} \cap H = \text{conv}\left(\bigcup_{r \in \mathscr{R}_0} r\right) \cap H$.

Lemma 2.5.1. *$\mathscr{R}_0$ is the set of rays in the boundary of $\overline{\mathscr{Z}}$.*

Proof. First, a ray $r \in \mathscr{R}_0$ is isotropic by Proposition 2.4.1, and thus it is in $\pm\partial(\overline{\mathscr{L}})) = \pm\partial(\overline{\mathscr{Z}})$ due to the description of $\mathscr{L}$ above. However, r must also be in $\overline{\mathscr{Z}}$, so $r \in \partial(\overline{\mathscr{Z}})$.

For the reverse implication, the statements above imply it is enough to show that the result is true inside of H, that is the convex hull of limit points of (images of) roots in H includes the entire boundary sphere of $\left(\overline{\mathscr{Z}} = \overline{\mathscr{L}}\right) \cap H$. We let the boundary of $\overline{\mathscr{Z}} \cap H$ be denoted by $\mathcal{S} := \partial\hat{\tau}(\overline{\mathscr{Z}})$, and we let $A := \hat{\tau}\left(\text{conv}\left(\bigcup_{r \in \mathscr{R}_0} r\right)\right)$. Then A is the convex hull of the points $\bigcup_{r\in\mathscr{R}_0}\hat{\tau}(r)$. But then the statement is clear since $\mathcal{S}$ is a sphere and is contained in A since $\overline{\mathscr{Z}}\cap H = \operatorname{conv}\left(\bigcup_{r\in\mathscr{R}_0} r\right)\cap H$. So every point in $\mathcal{S}$ must also be in A or else we would not have that the boundary of A is a sphere (which is true since (W,S) is hyperbolic). ☐

2.6. **Universal Coxeter Systems.** Suppose we have a hyperbolic Coxeter system (W,S) as in section 2.5. Then with the same setup there, we know that $\overline{\mathscr{Z}} = \overline{\mathscr{L}}$ and intersecting with H we get that $\mathcal{S} = \partial\hat{\tau}(\overline{\mathscr{Z}})$ is a sphere in H.

Lemma 2.6.1. *Suppose that $x, y \in \mathcal{S}$. Then there exist $\alpha_x, \alpha_y \in \Phi^+$ such that $\tau(\alpha_x), \tau(\alpha_y) \in \hat{\tau}\left(\mathscr{R}_+\right)$ are sufficiently close to x and y respectively; furthermore, α_x and α_y satisfy $(\alpha_x, \alpha_y) < -1$. In particular, if $S' := \{s_{\alpha_x}, s_{\alpha_y}\}$ then (W', S') is an indefinite infinite dihedral reflection subgroup of (W,S) and $\chi(W') = S'$.*

Proof. Since $x, y \in \mathcal{S}$ then x and y are limit points of $\hat{\tau}(\mathscr{R}_+)$ and therefore we can find (images of) positive roots arbitrarily close to x and y. Thus, we pick positive roots $\alpha_x \in \Phi^+$ and $\alpha_y \in \Phi^+$ such that $\tau(\alpha_x)$ and $\tau(\alpha_y)$ are close enough to x and y respectively to ensure that $\{a\tau(\alpha_x) + b\tau(\alpha_y) |\ a, b \in \mathbb{R}_{>0}\} \cap (\mathscr{Z}\cap H) \neq \emptyset$. Again, this is possible since $\mathcal{S}$ is a sphere and x and y are two points $(x \neq y)$ on the sphere and so the relative interior of the line connecting x and y must be included in the relative interior of the disk bounded by the sphere which is $\mathscr{Z}\cap H$. Now, we pick a point in the interior of the imaginary cone, $v = a\tau(\alpha_x) + b\tau(\alpha_y) \in \operatorname{int}(\mathscr{Z}\cap H) \subset \operatorname{int}(\mathscr{Z})$ with $a, b \in \mathbb{R}_{>0}$. Then $(v,v) < 0$ necessarily and so $(-,-)$ restricted to $\operatorname{Span}\{\alpha_x, \alpha_y\}$

cannot be of finite type or affine type. Thus, (W', S') cannot be a finite dihedral subgroup. Therefore, we see that $|(\alpha_x, \alpha_y)| > 1$. Now if $(\alpha_x, \alpha_y) > 1$ then by [3, Proposition 2.2] (and without loss of generality) we must have α_x dominates α_y. However, this is impossible since, by investigation of dominance in infinite dihedral groups ([8]), we know that α_x can dominate α_y only if $[\alpha_x, \alpha_y] \cap \mathscr{Z}_{W'} = \emptyset$, which is not the case here. Thus, $(\alpha_x, \alpha_y) < -1$ and so $S' = \{s_{\alpha_x}, s_{\alpha_y}\}$ is the canonical set of generators for the infinite dihedral group generated by S', (W', S'). $\square$

Now, by Lemma 2.2.1 recall that for arbitrary m we can find m points $\{x_1, ..., x_m\} \subset \mathcal{S}$ such that $P_m := \text{conv}\{x_1, ..., x_m\}$ along with the property that $d_K(P_m, \mathcal{S}) \to 0$ as $m \to \infty$. Then, by Lemma 2.6.1 above, since m is finite, we can find $\alpha_{x_i} \in \Phi^+$ with $\tau(\alpha_{x_i})$ sufficiently close to x_i such that $S'_{i,j} := \{s_{\alpha_{x_i}}, s_{\alpha_{x_j}}\}$ is the set of canonical generators for the (infinite) dihedral group it generates, $(W'_{i,j}, S'_{i,j})$. Now, let $S' := \{s_{\alpha_{x_1}}, ..., s_{\alpha_{x_m}}\}$ and W' be the reflection subgroup generated by S'.

Proposition 2.6.2. *The subsystem (W', S') described in the previous paragraph is a universal Coxeter system with $\chi(W') = S'$.*

Proof. Consider the Coxeter system (W', S'). Then by [10, Proposition 3.5], $S' = \chi(W')$ if and only if $\{s, s'\} = \chi(\langle s, s' \rangle)$ for all $s, s' \in S'$ and this follows from Lemma 2.6.1. Finally, since $m(s, s') = \infty$ for all $s \neq s' \in S'$ then (W', S') is a universal Coxeter system. $\square$

2.7. **Large Reflection Subgroups.** With the terminology as in section 2.5, we have H containing P. Thus, we can view $\hat{\tau}(\overline{\mathscr{Z}})$ as a subset of P. Additionally, we can view any subset of $\mathscr{R}_\Pi$ as a subset of P.

Thus, we will abuse notation by defining d_K on any compact subset of $\mathscr{R}_\Pi$ in the following way: for $X, Y \subset \mathscr{R}_\Pi$, $d_K(X,Y) = d_K(\hat{\tau}(X), \hat{\tau}(Y))$. We put the previous results together to obtain the following result.

Proposition 2.7.1. *Suppose (W,S) is hyperbolic. There exists a sequence of finite rank, universal reflection subgroups, (W'_m, S'_m), of W with $d_K(\mathbb{R}_{\geq 0}\Pi_{W'_m}, \overline{\mathscr{Z}}) \to 0$ as $m \to \infty$ and $d_K(\overline{\mathscr{Z}_{W'_m}}, \overline{\mathscr{Z}}) \to 0$ as $m \to \infty$.*

Proof. Let $\epsilon > 0$ be given. Then we can pick m large enough so that $d_K(P_m, \mathcal{S}) < \epsilon/2$ by Lemma 2.2.1 where $P_m = \operatorname{conv}\{x_1, ..., x_m\}$ for some $\{x_1, ..., x_m\} \subset \mathcal{S}$. Thus, $d_K(P_m, \overline{\mathscr{Z}} \cap H) < \epsilon/2$. According to Lemma 2.6.2 and the discussion before it, we can choose $\{\alpha_{x_1}, ..., \alpha_{x_m}\} \subset \Phi^+$ so that $\tau(\alpha_{x_i})$ is sufficiently close to x_i (in particular so that $d_K(\tau(\alpha_{x_i}), x_i) < \epsilon/2$) with the property that (W'_m, S'_m) is a universal Coxeter system where $S'_m := \{s_{\alpha_{x_1}}, ..., s_{\alpha_{x_m}}\}$ and $W'_m = \langle S'_m \rangle$. Now, let $Q_m = \operatorname{conv}\{\tau(\alpha_{x_1}), ..., \tau(\alpha_{x_m})\}$. Then $Q_m = \mathbb{R}_{\geq 0}\Pi_{W'_m} \cap H$ and $d_K(Q_m, P_m) < \epsilon/2$ since $d_K(\tau(\alpha_{x_i}), x_i) < \epsilon/2$ (by assumption above). Therefore, we see that

$$\begin{aligned} d_K(\mathbb{R}_{\geq 0}\Pi_{W'}, \overline{\mathscr{Z}}) &= d_K(Q_m, \overline{\mathscr{Z}} \cap H) \\ &\leq d_K(Q_m, P_m) + d_K(P_m, \overline{\mathscr{Z}} \cap H) \\ &= d_K(Q_m, P_m) + d_K(P_m, \mathcal{S}) < \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

as desired. It remains to show that we can take $d_K(\overline{\mathscr{Z}_{W'_m}}, \overline{\mathscr{Z}}) \to 0$ also.

We temporarily fix $i \neq j$. Then since $(\alpha_{x_i}, \alpha_{x_j}) < -1$, we have

$$\begin{aligned}\overline{\mathscr{Z}}_{\langle s_{\alpha_{x_i}}, s_{\alpha_{x_j}} \rangle} &= \{z \in \mathbb{R}_{\geq 0}\alpha_{x_i} + R_{\geq 0}\alpha_{x_j} |\ (z,z) \leq 0\} \\ &= \mathbb{R}_{\geq 0}\beta_{x_i,x_j} + \mathbb{R}_{\geq 0}\beta_{x_j,x_i}\end{aligned}$$

(which is a two-dimensional cone) for some isotropic linearly independent vectors β_{x_i,x_j} and β_{x_j,x_i}. We choose our previous notation so that $\beta_{x_i,x_j} \in \mathbb{R}_{\geq 0}\alpha_{x_i} + \mathbb{R}_{\geq 0}\beta_{x_j,x_i}$ and $\beta_{x_j,x_i} \in \mathbb{R}_{\geq 0}\beta_{x_i,x_j} + \mathbb{R}_{\geq 0}\alpha_{x_j}$.

Then, from [8], since $\langle s_{\alpha_{x_i}}, s_{\alpha_{x_j}} \rangle$ is dihedral, we have

$$\overline{\mathscr{Z}}_{\langle s_{\alpha_{x_i}}, s_{\alpha_{x_j}} \rangle} = \overline{\mathscr{Z}} \cap \left(\mathbb{R}_{\geq 0}\alpha_{x_i} + \mathbb{R}_{\geq 0}\alpha_{x_j}\right).$$

Thus, $\tau(\beta_{x_i,x_j})$ and $\tau(\beta_{x_j,x_i})$ are the points given by the intersection

$$[\tau(\alpha_{x_i}), \tau(\alpha_{x_j})] \cap \mathcal{S} = \{\tau(\beta_{x_i,x_j}), \tau(\beta_{x_j,x_i})\}.$$

This implies that we can choose α_{x_i} and α_{x_j} so that all of the following hold:

$$\begin{aligned} &d_K(\tau(\alpha_{x_i}), x_i) < \epsilon/4 \qquad && d_K(\tau(\alpha_{x_j}), x_j) < \epsilon/4 \\ &d_K(\tau(\beta_{x_i,x_j}), x_i) < \epsilon/4 \qquad && d_K(\tau(\beta_{x_j,x_i}), x_j) < \epsilon/4. \end{aligned} \tag{2.3}$$

Next, by Corollary 2.4.3, we know that

$$\overline{\mathscr{Z}}_{\langle s_{\alpha_{x_i}}, s_{\alpha_{x_j}} \rangle} \subseteq \overline{\mathscr{Z}}_{W'_m} \subseteq \overline{\mathscr{Z}}$$

for all pairs $i \neq j$. Each of these are convex cones, and so we also have

$$\sum_{i \neq j} \mathbb{R}_{\geq 0}\left(\mathbb{R}_{\geq 0}\beta_{x_i,x_j} + \mathbb{R}_{\geq 0}\beta_{x_j,x_i}\right) \subseteq \overline{\mathscr{Z}}_{W'_m} \subseteq \overline{\mathscr{Z}}. \tag{2.4}$$

Now, let $\epsilon > 0$. We choose m large enough and $\alpha_{x_1}, ..., \alpha_{x_m}$ sufficiently close to $x_1, ..., x_m$ respectively so that (2.3) holds and so that $d_K(\mathbb{R}_{\geq 0}\Pi_{W'_m}, \overline{\mathscr{Z}}) < \epsilon/2$ where $W'_m = \langle s_{\alpha_{x_1}}, ..., s_{\alpha_{x_m}} \rangle$ (which is possible by the first part of the proposition). Then (2.3) implies that $d_K(\alpha_{x_i}, \beta_{x_i,x_j}) < \epsilon/2$ for all $j \neq i$ so that

$$d_K([\tau(\beta_{x_i,x_j}), \tau(\beta_{x_j,x_i})], [\tau(\alpha_{x_i}), \tau(\alpha_{x_j})]) < \epsilon/2$$

and thus

$$d_K\left(\sum_{i\neq j}\left(\mathbb{R}_{\geq 0}\beta_{x_i,x_j} + \mathbb{R}_{\geq 0}\beta_{x_j,x_i}\right), \sum_{k=1}^{m}\mathbb{R}_{\geq 0}\alpha_{x_k}\right) < \frac{\epsilon}{2}.$$

Together, since $\sum_{k=1}^{m}\mathbb{R}_{\geq 0}\alpha_{x_k} = \mathbb{R}_{\geq 0}\Pi_{W'_m}$, we get that

$$d_K\left(\sum_{i\neq j}\left(\mathbb{R}_{\geq 0}\beta_{x_i,x_j} + \mathbb{R}_{\geq 0}\beta_{x_j,x_i}\right), \overline{\mathscr{Z}}\right) \leq$$
$$d_K\left(\sum_{i\neq j}\left(\mathbb{R}_{\geq 0}\beta_{x_i,x_j} + \mathbb{R}_{\geq 0}\beta_{x_j,x_i}\right), \sum_{k=1}^{m}\mathbb{R}_{\geq 0}\alpha_{x_k}\right) + d_K(\mathbb{R}_{\geq 0}\Pi_{W'_m}, \overline{\mathscr{Z}}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Then, equation (2.4) implies that $d_K(\overline{\mathscr{Z}}_{W'_m}, \overline{\mathscr{Z}}) < \epsilon$ as well, finishing the proof. □

The next theorem follows from the previous propositions and [11].

Theorem 2.7.2. *Let (W, S) be a finite rank, irreducible, infinite, non-affine Coxeter system. Then, for any $m \in \mathbb{N}$, W contains a reflection subgroup W' with $|\chi(W')| = m$ such that $(W', \chi(W'))$ is a universal Coxeter system.*

Proof. Let $m \in \mathbb{N}$. By Theorem 2.2 of [11], (W, S) contains a hyperbolic parabolic subsystem (W_J, J). By Proposition 2.6.2, we have that (W_J, J) contains a universal Coxeter system (W', S') of rank m and thus (W, S) also contains (W', S') proving the theorem. $\square$

3. Exponential Growth

3.1. **Growth in Coxeter Groups.** For any finitely generated group, we can define the growth function of the group by determining the number of elements of the group of all lengths (with respect to the generating set). In [4], de la Harpe demonstrates that irreducible, infinite, non-affine Coxeter systems have what is known as exponential growth by showing that any such Coxeter system must contain a free non-abelian subgroup. Also, in [15], Margulis and Vinberg prove the stronger result that such a W must contain a finite index subgroup which surjects onto a non-abelian free group. Finite and affine Coxeter systems are known to have polynomial growth.

Even more recently, in [16], Viswanath uses an alternate method to show that any irreducible, infinite, non-affine *simply-laced* Coxeter system has exponential growth. He also obtains the stronger result that for such a system (W, S) and any $J \subsetneq S$ the quotient W/W_J also has exponential growth. His method involves finding a particular universal rank 3 reflection subgroup inside of (W, S). In [16, Remark 3], the author notes that it would be interesting to know if this result holds in the non-simply laced case as well. We intend to answer this question using our results from the previous section in order to modify Viswanath's argument.

3.2. **Growth Types.** We first introduce the basic notions of growth types in finitely generated Coxeter groups. Let (W, S) be a finitely generated Coxeter system. We follow [5] or [16] for general terminology and results.

Definition 3.2.1. Let $\mathbf{a} := (a_k)_{k\geq 0}$ be a sequence of non-decreasing natural numbers. We define the *exponential growth rate* of the sequence to be $\omega(\mathbf{a}) := \limsup_{k\to\infty} a_k^{1/k}$.

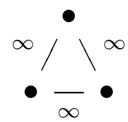
For each $k \in \mathbb{N}$, we define a subset $W_{\leq k} := \{w \in W|\ l(w) \leq k\}$. Then, for any subset $F \subseteq W$ with $1 \in F$, we define $F_{\leq k} := F \cap W_{\leq k}$. Finally, for any $k \in \mathbb{N}$ and $F \subseteq W$, we define the number $\gamma_{F,k} := |F_{\leq k}|$ and the sequence $\gamma_F := (\gamma_{F,k})_{k\geq 0}$. Then, let $\omega(F) := \omega(\gamma_F) = \limsup_{k\to\infty} \gamma_{F,k}^{1/k}$.

Then, we say that a subset F has exponential growth if $\omega(F) > 1$ and subexponential growth otherwise. If a subset F has subexponential growth, and there is $C \in \mathbb{R}_{>0}$ and $d \in \mathbb{Z}_{\geq 0}$ with $\gamma_{F,k} \leq Ck^d$ for all $k \geq 0$, then we say F has *polynomial growth*. If F is of subexponential growth and not of polynomial growth, F has *intermediate growth*.

We note that if $P(F, W) = \sum_{w\in F} u^{l(w)} = \sum_{k\geq 0} a_k u^k$ is the Poincaré series of F, then the coefficients $a_k = \gamma_{F,k} - \gamma_{F,k-1}$ for $k \geq 1$ and $a_0 = \gamma_{F,0}$.

3.3. **Properties of $W_{(3)}$.** The proof that quotients have exponential growth requires some properties of the universal Coxeter system $W_{(3)} =$

$\langle a, b, c |\ a^2 = b^2 = c^2 = 1\rangle$ with Coxeter diagram



These properties are all known and listed in [16], but we remind the reader here for completeness.

$W_{(3)}$ has the following properties:

(1) The Poincaré series of $W_{(3)}$ is $P(W_{(3)}) = \frac{1+q}{1-2q}$.

(2) Each $w \in W_{(3)}$ has a unique reduced expression.

(3) The conjugacy classes of a, b, and c are all distinct.

(4) For $x \neq y \in W_{(3)}/(W_{(3)})_{\{a\}} =: W_{(3)}^{\{a\}}$, we know that $xax^{-1} \neq yay^{-1}$. Also, $P(W_{(3)}^{\{a\}}, W_{(3)}) = \frac{P(W_{(3)})}{1+q} = \frac{1}{1-2q}$ so there are exactly 2^k elements $x \in W_{(3)}^{\{a\}}$ with $l(x) = k$. For each such x, we have $l(xax^{-1}) \leq 2l(x) + 1 = 2k + 1$.

(5) Property 4 implies that for $k \geq 0$ there are more than 2^k elements $t \in \{xax^{-1} |\ x \in W_{(3)}^{\{a\}}\}$ with $l(t) \leq 2k + 1$.

(6) For any reflection $t \in \{xax^{-1} |\ x \in W_{(3)}^{\{a\}}\}$, properties 2 and 3 imply that a appears in any reduced expression for t.

3.4. **Parabolic Quotients.** With use the knowledge of $W_{(3)}$, we prove the following generalization of [16, Theorem 1]. Recall that for this entire paper, (W, S) is finite rank.

Theorem 3.4.1. *Let (W, S) be a finite-rank, irreducible, infinite, non-affine Coxeter system. Then for all $J \subsetneq S$, $W^J = W/W_J$ has exponential growth.*

Our proof is similar to the one found in [16], however, we need to make some changes so that proof works for all Coxeter systems (as opposed to only simply-laced Coxeter systems). We first recall an important result due to Deodhar, [6].

Theorem 3.4.2. *Let (W, S) be a Coxeter system, T the set of reflections, and $J \subseteq S$. If $t_1, t_2 \in T \setminus W_J$ with $t_1 \neq t_2$, then $t_1 W_J \neq t_2 W_J$, that is, distinct reflections in $T \setminus W_J$ lie in distinct left cosets of W_J.*

As is demonstrated in [16], to prove Theorem 3.4.1, we need to prove the following proposition.

Proposition 3.4.3. *Let (W, S) be an irreducible, infinite, non-affine Coxeter system and let $J \subsetneq S$. Then there exists $M \in \mathbb{N}$ such that for all $k \geq 0$ there are at least 2^k elements $t \in T \setminus W_J$ such that $l(t) \leq M(2k+1)$.*

If this proposition holds, Theorem 3.4.1 holds by the same argument given in [16].

3.5. **Proof of Proposition 3.4.3.**

Proof. For all $x \in W$, any reduced expression for x uses the same simple reflections. We write

$$\operatorname{supp}(x) = \{r \in S \mid r \text{ appears in all reduced expression for } x\}.$$

It is also well known that $x \in W_J$ if and only if $\operatorname{supp}(x) \subseteq J$.

Now, let (W, S) be an irreducible, infinite, non-affine Coxeter system and let $s' \notin J$. According to Theorem 2.7.2, (W, S) contains a reflection subgroup W' such that $W' \cong W_{(3)}$. Suppose $\chi(W') = \{t_1, t_2, t_3\}$. Without loss of generality, we may assume that $s' \in \mathrm{supp}(t_1)$. Indeed, if $s' \notin \mathrm{supp}(t_i)$ for all i, then let $Z \subset S$ be the smallest subset such that $W' \subset W_Z$. Then, take a minimal path $s' = s_n - s_{n-1} - \cdots - s_1 - z$ in the Coxeter diagram with $s_i \notin Z$ for all i and $z \in Z$; let the word corresponding to the path be $x = s_1 \cdots s_n \in W$, and by reordering we assume $z \in \mathrm{supp}(t_1)$. Now, we define a sequence of reflections in the following way. Let $r_0 = t_1$ and $r_i = s_i \cdots s_1 t_1 s_1 \cdots s_i$ for all $i \geq 1$. Let $\alpha_{r_i} \in \Phi^+$ be the root corresponding to r_i. Now, since $(\{s_1, ..., s_{i-1}\} \cup Z) \cap (\{s_i\}) = \emptyset$ for all $i \geq 1$ and s_i is connected to s_{i-1}, we have $(\alpha_{r_{i-1}}, \alpha_{s_i}) < 0$ for all i and so $l(r_i) = l(r_{i-1}) + 2$ for all $i \geq 1$. This implies that $l(x^{-1}t_1x) = 2l(x) + l(t_1)$; hence, $s' = s_n \in \mathrm{supp}(x^{-1}tx)$. Then, $xW'x^{-1}$ is a Coxeter system, isomorphic to $W_{(3)}$ with $\chi(xW'x^{-1}) = \{xtx^{-1}, xt_2x^{-1}, xt_3x^{-1}\} = \{t'_1, t'_2, t'_3\}$ with $s' \in \mathrm{supp}(t'_1)$ as required.

Next, by the properties listed in Section 3.3, we know that there are at least 2^k elements $t \in W' \cap T$ such that $l_{(W', \chi(W'))}(t) \leq 2k+1$ such that $t_1 \in \mathrm{supp}(t)$. Define $M := \max\{l(t_i), l(t_2), l(t_3)\}$. Then, we have $l(w) \leq M(l_{(W', \chi(W'))}(w))$ for all $w \in W'$. This implies that for each $t \in W' \cap T$ above, we have $l(t) \leq M(2k+1)$.

For each $t \in W' \cap T$ described above, t_1 appears in a reduced expression for t; in terms of roots this means $\alpha_t = \sum_{i=1}^{3} c_i \alpha_{t_i}$ with $c_1 > 0$. Since $s' \in \mathrm{supp}(t_1)$, we have $\alpha_{t_1} = \sum_{s \in S} c_s \alpha_s$ with $c_{s'} > 0$. Therefore, it follows that $\alpha_t = \sum_{s \in S} d_s \alpha_s$ with $d_{s'} > 0$ so that $s' \in \mathrm{supp}(t)$. Thus, we have $t \notin W_J$. This implies that $t \in T \setminus W_J$. Therefore, by Theorem

3.4.2, we get that there are at least 2^k elements $t \in T \setminus W_J$ such that $l(t) \leq M(2k+1)$. The proposition follows. □

References

[1] Alexander Barvinok. *A course in convexity*, volume 54 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.

[2] Anders Björner and Francesco Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, New York, 2005.

[3] Brigitte Brink. The set of dominance-minimal roots. *J. Algebra*, 206(2):371–412, 1998.

[4] Pierre de la Harpe. Groupes de Coxeter infinis non affines. *Exposition. Math.*, 5(1):91–96, 1987.

[5] Pierre de la Harpe. *Topics in geometric group theory*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000.

[6] Vinay V. Deodhar. On the root system of a Coxeter group. *Comm. Algebra*, 10(6):611–630, 1982.

[7] Vinay V. Deodhar. A note on subgroups generated by reflections in Coxeter groups. *Arch. Math. (Basel)*, 53(6):543–546, 1989.

[8] Matthew Dyer. Imaginary cone and dominance order in Coxeter groups. in preparation.

[9] Matthew Dyer. Imaginary cone and reflection subgroups of Coxeter groups. in preparation.

[10] Matthew Dyer. Reflection subgroups of Coxeter systems. *J. Algebra*, 135(1):57–73, 1990.

[11] Tom Edgar. A case-free characterization of hyperbolic Coxeter systems. to appear in J. Group Theory. Available online: 10.1515/JGT.2011.077.

[12] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[13] Victor G. Kac. *Infinite-dimensional Lie algebras*. Cambridge University Press, Cambridge, third edition, 1990.

[14] K. Kuratowski. *Topology. Vol. I*. New edition, revised and augmented. Translated from the French by J. Jaworowski. Academic Press, New York, 1966.

[15] G. A. Margulis and È. B. Vinberg. Some linear groups virtually having a free quotient. *J. Lie Theory*, 10(1):171–180, 2000.

[16] Sankaran Viswanath. On growth types of quotients of Coxeter groups by parabolic subgroups. *Comm. Algebra*, 36(2):796–805, 2008.

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