Wyld Weyl and Twisted Bruhat: Partial Orders on Groups

Tom Edgar

Department of Mathematics
University of Notre Dame
Notre Dame, IN

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Introduction

Coxeter Systems (Coxeter Groups)

Classical Bruhat Order

Cocyles and ”twisted Bruhat orders”

Current Research

Coxeter Groups

Setup: Let $S = \{s_1, s_2, ..., s_n\}$ be a finite set. Suppose that we have a function $m : S \times S \to \mathbb{Z}_{\geq 1} \cup \infty$ satisfying $m(s_i, s_j) = m(s_j, s_i)$, $m(s_i, s_i) = 1$ and $m(s_i, s_j) \geq 2$ for $i \neq j$.

Definition

Let W be the group generated by S subject *only* to the relations $(s_i s_j)^{m(s_i, s_j)} = 1$ ($m(s_i, s_j) = \infty$ means there is no relation). In this case, we say that (W, S) is a **Coxeter system** (and we say W is a Coxeter group).

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Implications

- W consists of "words" in S.
- For $s \in S$, $m(s,s) = 1$ means $ss = s^2 = 1$.
- For $r, s \in S$, $m(r,s) = k$ means

$$(rs)^k = \underbrace{rs \cdot rs \cdots rs \cdot rs}_{k \text{ times}} = 1$$

- This implies the "braid relations:"

$$\underbrace{rsr\ldots}_{k} = \underbrace{srs\ldots}_{k}$$

- For example, $m(r,s) = 2$ means $rs = sr$.

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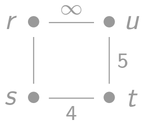
Organizing Data

Definition

A good way to organize this data is by using the so-called **Coxeter graph**. We construct the graph as follows:

1. Let the set of vertices be S.
2. We connect two vertices s_i and s_j if $m(s_i, s_j) \geq 3$.
3. We label the edge with $m(s_i, s_j)$ if $m(s_i, s_j) \geq 4$.

Example



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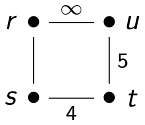

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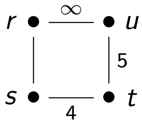

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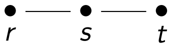

S_3: A Detailed Example

Standard Notation

1. Generators: (12) and (23)
2. $(12)(12) = 1$
   $(23)(23) = 1$
3. $(12)(23)(12) = (13)$
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Coxeter Group Notation

1. $r = (12)$ and $s = (23)$
2. $r^2 = 1$
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3. $rsr = srs$
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So, $S_3 = \langle r, s | \ r^2 = s^2 = (rs)^3 = 1 \rangle$

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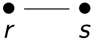

Length

In S_3, any word can be written with 3 letters or less:

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Definition

For $w \in W$, if $w = s_{i_1} \dots s_{i_k}$ is written as short as possible, we say that this is a **reduced expression** for w and we say that w has **length** k. We write $\ell(w) = k$.

- $\ell : W \to \mathbb{N}$
- $\ell(1) = 0$
- w can have multiple reduced expressions
  - All have same length
  - e.g. $\ell(rsr) = \ell(srs)$ in S_3

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Posets

Definition

Given a set P, we say a binary relation $\leq$ is a **partial order** on P if it satisfies three properties:

- Reflexivity: $a \leq a$ for all $a \in P$.
- Antisymmetry: If $a \leq b$ and $b \leq a$ then $a = b$.
- Transitivity: If $a \leq b$ and $b \leq c$ then $a \leq c$.

If we loosen the antisymmetry requirement, we call $\leq$ a **pre-order**. A set with a partial order is called a **poset**.

Bruhat Order

We turn W into a poset in the following way:

- Fix an arbitrary reduced expression for $w \in W$. We say $v \leq w$ if v can be written as some subexpression of this reduced expression for w.
- 1 is the minimum element; poset is graded by length.
- For W finite, there is a unique maximum element.
- It is clearly a poset: reflexivity, antisymmetry, and transitivity are all clear via this definition.

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The partial order defined above is called the **Bruhat order** on a Coxeter Group.

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Recall that S_3 has 6 elements: $1, r, s, rs, sr, rsr = srs$.

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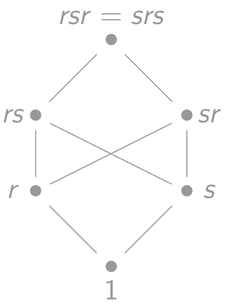

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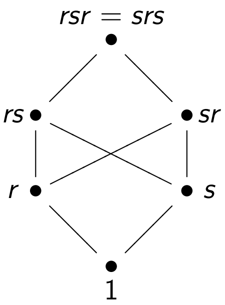
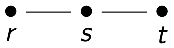
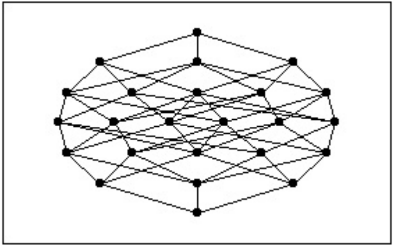


Some Pictures of Bruhat orders

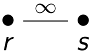
Example (S_4)

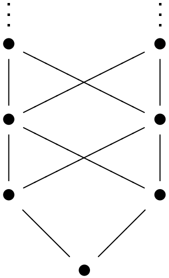




http://www.haverford.edu/math/cgreene/posets/posetgallery.html

Example (D_∞)





A More Natural Definition (Cocycles)

Setup: Suppose (W, S) is a Coxeter System.

- $T = \bigcup_{w \in W} wSw^{-1}$, "reflections" of W.
- $\mathcal{P}(T)$=power set of T
  - Forms a group under symmetric difference:

$$A + B = (A \cup B) \setminus (A \cap B)$$

- W acts on $\mathcal{P}(T)$ via conjugation:

$$wAw^{-1} \subseteq T \text{ for all } A \subseteq T$$

Definition

Using this action, a **cocycle** on W is a map $f : W \to \mathcal{P}(T)$ satisfying $f(xy) = f(x) + xf(y)x^{-1}$ for all $x, y \in W$.

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Using this action, a **cocycle** on W is a map $f : W \to \mathcal{P}(T)$ satisfying $f(xy) = f(x) + xf(y)x^{-1}$ for all $x, y \in W$.

A More Natural Definition (Cocycles)

Setup: Suppose (W, S) is a Coxeter System.

- $T = \bigcup_{w \in W} wSw^{-1}$, "reflections" of W.
- $\mathcal{P}(T)$=power set of T
  - Forms a group under symmetric difference:
$$A + B = (A \cup B) \setminus (A \cap B)$$
- W acts on $\mathcal{P}(T)$ via conjugation:
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A More Natural Definition (Bruhat Graph)

Definition

$N : W \to \mathcal{P}(T)$ given by $N(w) := \{t \in T | \ \ell(tw) < \ell(w)\}$ is called the **reflection cocycle**.

We can "twist" the conjugation action with this cocycle to get a new action: For $A \in \mathcal{P}(T)$ and $w \in W$

$$w \cdot A = N(w) + wAw^{-1}$$

Definition

Let Ω_W to be the directed graph given by the following:

1. The vertices are the elements of W.
2. The edge set is $E_W = \{(tw, w) | \ t \in T \text{and } t \in w \cdot \emptyset\}$.

Then, the Bruhat order can be restated as $v \leq w$ iff there is a path from v to w in Ω_W.

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Generalization

Definition

For any $A \in \mathcal{P}(T)$, let $\Omega_{(W,A)}$ be the directed graph given by the following:

1. The vertices are the elements of W.
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Definition

We define a pre-order $\leq_A$ on W by $v \leq_A w$ if there is a path from v to w in $\Omega_{(W,A)}$ for $v, w \in W$.

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Question

When is $\leq_A$ a partial order? In other words, for what $A \in \mathcal{P}(T)$ is $\leq_A$ antisymmetric?

Subsets of T

Definition

A **dihedral reflection subgroup** is a subgroup of W generated by 2 elements of T. Any dihedral reflection subgroup has a canonical set of generators.

- D is a dihedral reflection subgroup generated by r and s.

$$T \cap D = \{r, rsr, rsrsr, ..., srsrs, srs, s\}$$

Definition

We say total order $\prec$ on T is a **reflection order** if one of

- $r \prec rsr \prec \cdots \prec srs \prec s$
- $s \prec srs \prec \cdots \prec rsr \prec r$

is true for any dihedral reflection subgroup D of W.

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The Main Result

Definition

A subset $A \subseteq T$ is an **initial section of a reflection order** (i.s.o.r.o) if there is some reflection order $\prec$ with the property that $a \prec b$ for all $a \in A$ and $b \in T \setminus A$.

Now, we define

$$\mathbf{A}_{(W,S)} = \{A \subseteq T |\ A \cap D \text{ is an i.s.o.r.o on } D\}$$

where D ranges over all dihedral reflection subgroups of W.

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where D ranges over all dihedral reflection subgroups of W.

Theorem

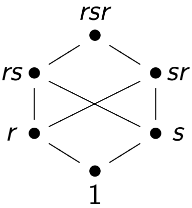
Let (W, S) be a Coxeter system with reflections T and let $A \subseteq T$. $\leq_A$ is a partial order if and only if $A \in \mathbf{A}_{(W,S)}$.

S_3: A Detailed Example

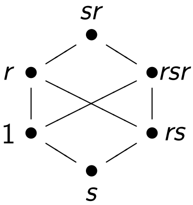
- S_3 is a dihedral group (and has no proper dihedral subgroups).
- $T = \{r, rsr, s\}$; $\mathcal{P}(T)$ has 8 elements.
- 2 possible reflection orders: $r \prec rsr \prec s$ or $s \prec rsr \prec r$.
- 6 initial sections: $\emptyset, \{r\}, \{s\}, \{r, rsr\}, \{s, rsr\}, T$.
- 2 non-initial sections: $\{r, s\}$ and $\{rsr\}$.
- 6 twisted Bruhat orders. (All isomorphic since S_3 is finite.)

S_3: A Detailed Example

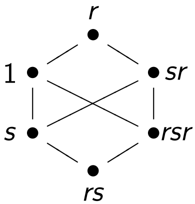
$A = \emptyset$



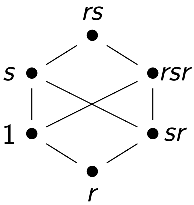
$A = \{s\}$



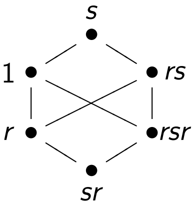
$A = \{s, rsr\}$



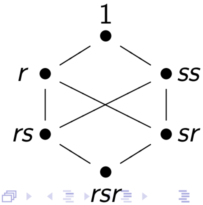
$A = \{r\}$



$A = \{r, rsr\}$



$A = T = \{r, rsr, s\}$



Current Research

- Representation Theory associated to twisted Bruhat orders:
  - Kazhdan-Lusztig Polynomials.
  - Positivity conjectures.
- Classify $H^1(W, \mathcal{P}(T))$ for Coxeter groups (Finite, Weyl, etc.).
- Do other cocycles lead to posets on W in a similar manner?
- Use cocycles to understand other properties of Coxeter groups.

Thanks!

Any Questions?

Bruhat

Weyl