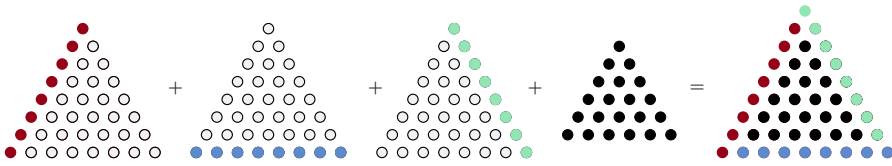


# Proof Without Words: A Recursion for Triangular Numbers and More

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**Theorem.** *The triangular numbers,  $t_n := 1 + 2 + 3 + \dots + n$ , satisfy the recursion  $t_n = 3t_{n-1} - 3t_{n-2} + t_{n-3}$ .*

*Proof.*



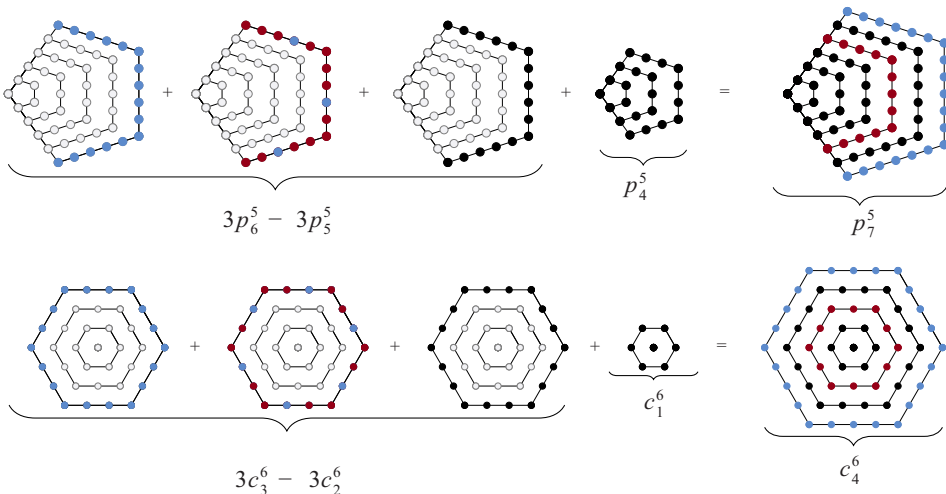
If  $a_n$  and  $b_n$  are sequences that both satisfy the same recurrence, any sequence  $s_n := ca_n + db_n$  satisfies the recurrence as well; so, the theorem provides a wordless proof for each of the following three statements (since  $z_n = 1$  satisfies the recurrence).

**Corollary 1.** *The  $k$ -gonal numbers, given by either  $p_n^k = \sum_{i=0}^{n-1} ((k-2) \cdot i + 1)$  or  $p_n^k = (k-3)t_{n-1} + t_n$ , satisfy the recursion  $p_n^k = 3p_{n-1}^k - 3p_{n-2}^k + p_{n-3}^k$ .*

**Corollary 2.** *The centered  $k$ -polygonal numbers, given by  $c_n^k = kt_{n-1} + 1$ , satisfy the recursion  $c_n^k = 3c_{n-1}^k - 3c_{n-2}^k + c_{n-3}^k$ .*

**Corollary 3.** *Any quadratic sequence  $f_n = an^2 + bn + c$  satisfies the recursion  $f_n = 3f_{n-1} - 3f_{n-2} + f_{n-3}$  since  $n^2 = t_n + t_{n-1}$  and  $n = t_n - t_{n-1}$ .*

Corollaries 1 and 2 can be visualized directly using diagrams analogous to the following for the pentagonal and centered hexagonal numbers.



**Summary.** We visually demonstrate a recurrence satisfied by the triangular numbers and hence all quadratic sequences including all types of polygonal numbers.

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### A New Theory of Strings

Yes, no one knows if atoms are real  
but they sure seem like they exist.  
After all, what are we made of  
beyond skin and bones and cells  
and maybe some other things like love.

So, a name for something tiny  
might as well be pieces of string,  
Which reminds me of a shiny harp!  
Yes, that's the kind of thing  
it could be, because when  
the harpist plucks, watch the strings.

They seem to vibrate with music,  
yes invisible sounds that penetrate  
thru our ears to our very souls,  
which, of course, no one can see,  
but we know exist, in reality,  
or theory?

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