Staircase Series

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Swain [5] uses what he refers to as "Gabriel's Staircase" to prove visually that $\sum_{k=1}^{\infty} kr^k = \frac{r}{(1-r)^2}$ for $0 < r < 1$. In this note, we demonstrate how to modify his diagram to provide recursive wordless computations of series of the form $\sum_{k=0}^{\infty} k^n r^k$ where $n \geq 1$, which we will thus refer to as *staircase series*. For instance, in Figure 1 we vary the "stair heights" according to successive odd integers, resulting in a visual proof that

$$\sum_{k=1}^{\infty} k^2 r^k = \frac{1}{1-r} \sum_{k=1}^{\infty} (2k-1) r^k.$$

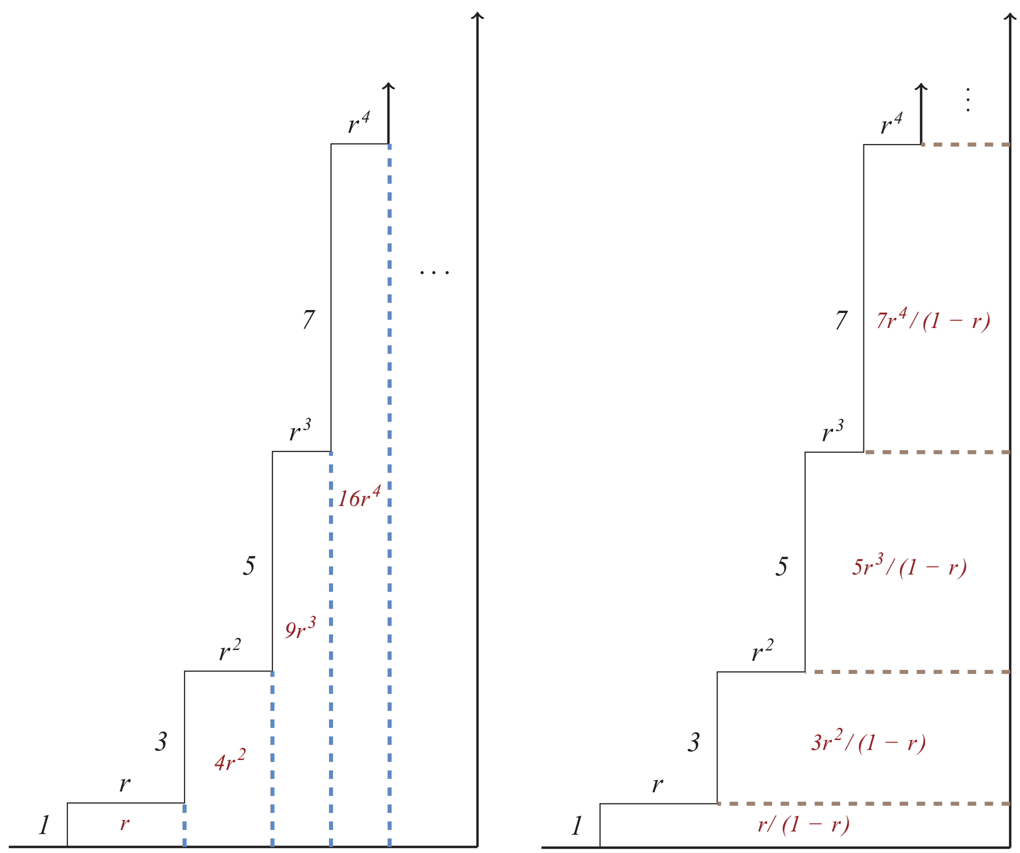

Figure 1 Staircase series with odd integer stair heights.

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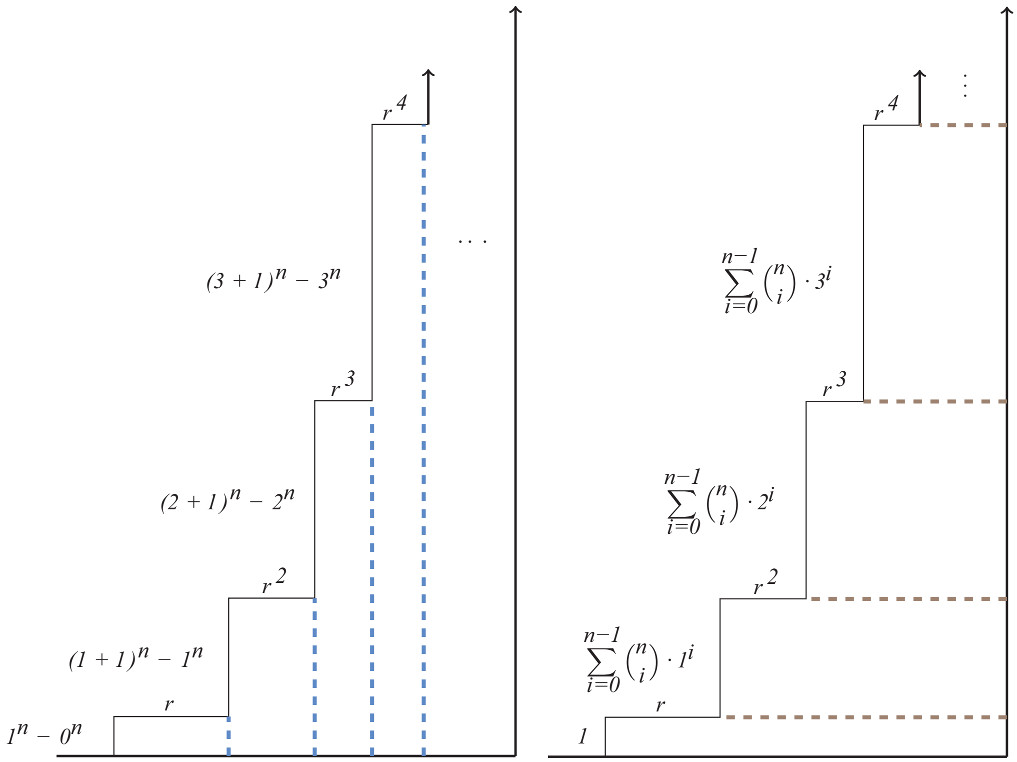

Figure 2 A general staircase series diagram.

Of course, if we couple this with Swain's result (and some elementary algebra) we obtain a visual proof that

$$\sum_{k=1}^{\infty} k^2 r^k = \frac{1}{1-r} \cdot \left(2 \cdot \frac{r}{(1-r)^2} - \frac{r}{1-r}\right) = \frac{r^2+r}{(1-r)^3}.$$

This extension of Swain's diagram works because every square, n^2, is the sum of the first consecutive n odd integers so that the difference of consecutive squares is an odd number (and every odd number is the difference of two consecutive squares). This realization about Figure 1 helps us see that we can further generalize Swain's diagram to provide a reduction formula for the series $\sum_{k=0}^{\infty} k^n r^k$, which can be found in [**2**] or [**3**]. To build this diagram, we simply label "stair heights" with the differences of successive powers of n, which we have done in Figure 2. According to the binomial theorem, differences of successive powers of n can be computed in terms of binomial coefficients. Consequently, investigation of Figure 2 leads to the following reduction formula:

$$\sum_{k=0}^{\infty} k^n r^k = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{n-1} \binom{n}{i} k^i \cdot \sum_{j=k+1}^{\infty} r^j \right) = \sum_{k=0}^{\infty} \left(\sum_{i=0}^{n-1} \binom{n}{i} k^i \cdot \frac{r^{k+1}}{1-r} \right).$$

Elementary algebraic manipulation (no derivatives needed) shows that for $n \geq 1$

$$\sum_{k=0}^{\infty} k^n r^k = \frac{r}{1-r} \cdot \sum_{i=0}^{n-1} \left(\binom{n}{i} \sum_{k=0}^{\infty} k^i r^k \right). \tag{1}$$

Exercise 1. Find a closed form for $\sum_{k=0}^{\infty} k^3 r^k$ in terms of r when $0 < r < 1$.

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 4 & & 1 & & & \\ & & 1 & & 11 & & 11 & & 1 & & \\ & 1 & & 26 & & 66 & & 26 & & 1 & \\ 1 & & 57 & & 302 & & 302 & & 57 & & 1 \end{array}$$

Figure 3 The triangle of Eulerian numbers up to row 5.

According to a variant of the Carlitz identity (see [1] or [4]), for $n \geq 1$, each staircase series has a closed form

$$\sum_{k=0}^{\infty} k^n r^k = \frac{r \cdot p_n(r)}{(1-r)^{n+1}},$$

where $p_n(r)$ is a polynomial in r of degree $n-1$. This identity was known to Euler; in fact the polynomials, $p_n(r)$, are known as the Eulerian polynomials, and their coefficients are given by the Eulerian numbers, which form a triangular array of numbers similar to the binomial coefficients. More precisely,

$$p_n(r) = \sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle r^i,$$

where $\left\langle {n \atop i} \right\rangle$ is entry i in row n of the triangle of Eulerian numbers.

The Eulerian number $\left\langle {n \atop i} \right\rangle$ counts the number of permutations of $\{1, \ldots, n\}$ with i descents, where a descent of a permutation, π, is a position j such that $\pi(j) > \pi(j+1)$. The triangle of Eulerian numbers can be obtained recursively as follows:

$$\left\langle {n \atop i} \right\rangle = (n-i)\left\langle {n-1 \atop i-1} \right\rangle + (i+1)\left\langle {n-1 \atop i} \right\rangle,$$

where $\left\langle {1 \atop 0} \right\rangle = 1$ and $\left\langle {n \atop i} \right\rangle = 0$ if $i \geq n$ or if $i < 0$. Rows 1 through 5 of the triangle of Eulerian numbers are shown in Figure 3. For instance, we see that $p_3(r) = r^2 + 4r + 1$, and thus the answer to Exercise 1 is

$$\sum_{k=0}^{\infty} k^3 r^k = \frac{r^3 + 4r^2 + r}{(1-r)^4}.$$

So, our visual computation of staircase series, and its production of equation (1), coupled with the Carlitz identity, allows us to see that

$$\begin{aligned} \frac{r \cdot p_n(r)}{(1-r)^{n+1}} = \sum_{k=0}^{\infty} k^n r^k &= \frac{r}{1-r} \cdot \sum_{i=0}^{n-1} \left(\binom{n}{i} \sum_{k=0}^{\infty} k^i r^k \right) \\ &= \frac{r}{1-r} \cdot \sum_{i=0}^{n-1} \left(\binom{n}{i} \frac{r \cdot p_i(r)}{(1-r)^{i+1}} \right), \end{aligned}$$

which leads to the following recursive construction of the Eulerian polynomials:

$$p_n(r) = \sum_{i=0}^{n-1} \left(\binom{n}{i} r(1-r)^{n-i-1} p_i(r) \right), \tag{2}$$

where here we must set $p_0(r) = \frac{1}{r}$ since the variation on the Carlitz identity does not hold for $n = 0$.

Exercise 2. Use Figure 3 and equation (2) to find $p_7(r)$, then find the sum of the series $\sum_{k=0}^{\infty} k^7 (\frac{1}{2})^k$.

While "proofs without words" are not formal proofs, the point of such diagrams is to provide a visual that helps one recreate the theorem and/or proof. The reduction formula given in equation (1) can quickly be reconstructed by remembering the diagram in Figure 2. This reduction formula provides a method of computing the sum of a variety of series without having to apply calculus techniques making it suitable for introducing students to series. Surprisingly at first, the same reduction formula also has many interesting connections to combinatorics, as can be seen in [**2**] and [**3**] as well as our included recursion for Eulerian polynomials.

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Summary. We provide a wordless calculation of the series $\sum_{i=0}^{\infty} k^2 r^k$ for any $0 < r < 1$. We extend this visual proof to give an algebraic reduction formula for $\sum_{i=0}^{\infty} k^n r^k$ where $n \geq 1$ is an integer, and we discuss a combinatorial consequence of this formula.

TOM EDGAR (MR Author ID: 821633) received his Ph.D. from the University of Notre Dame and is an associate professor at Pacific Lutheran University in Tacoma, WA. He enjoys helping his 3/2-year old son learn to explore the world and hopes that one day his son will learn how to climb staircase series.