REDUCED EXPRESSIONS IN SEMIDIRECT PRODUCTS OF COXETER GROUPS

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ABSTRACT. Any normal reflection subgroup $\widetilde{W}$ of a Coxeter system (W, S) is a factor in a semidirect product decomposition of W as described by Bonnafé and Dyer. Namely, S is the union of two subsets I and J such that no element of I is conjugate to an element of J, $\widetilde{W}$ is the subgroup generated by W_I-conjugates of elements of J, and W is the semidirect product of W_I by $\widetilde{W}$. This note describes the reduced expressions of elements of the form wxw^{-1} with $w \in W_I$ and $x \in W_J$ in terms of reduced expressions of x and a suitable element of W_I.

INTRODUCTION AND PRELIMINARIES

Following Bonnafé and Dyer, we consider a Coxeter system (W, S) in its internal semidirect product decomposition with respect to two subsets $I, J \subset S$ such that $I \cap J = \emptyset$ and $I \cup J = S$ and with the requirement that no element of I is conjugate to an element of J. The latter statement says that the vertex set of the Coxeter graph of (W, S) can be divided into two subsets with only edges with even label connecting the two. Let T represent the set of reflections of W. In [4] (see also [1]), it is shown that W is the semidirect product of W_I and the normal subgroup $\widetilde{W}$ generated by W_I-conjugates of elements of J. We observe that any normal reflection subgroup $\widetilde{W}$ of W arises from this type of decomposition of S. Then, we describe the reduced expressions of elements of the form wxw^{-1} where $w \in W_I$ and $x \in W_J$ in terms of reduced expressions of w and x. From this, we get a slightly stronger result describing reduced expressions of reflections of the form wtw^{-1} with $t \in W_J \cap T$ and $w \in W_I$. For general results about Coxeter systems consult [5].

1. STATEMENT OF MAIN RESULT

We begin with the same setup as [1]. We let (W, S) be a Coxeter system with $\ell : W \to \mathbb{N}$ the corresponding length function. Furthermore, assume that $S = I \cup J$ with $I \cap J = \emptyset$ and that no element of I is conjugate to an element of J. We denote by W_I and W_J the standard parabolic subgroups generated by I and J respectively. For any $K, L \subset S$, we let $W^K := W/W_K$ (respectively ${}^LW := W_L \backslash W$) be the set of shortest right coset representatives of W_K in W (respectively the set of shortest left coset representatives of W_L in W). Then we define ${}^LW^K := {}^LW \cap W^K$ to be the set of shortest representatives for the double cosets $W_L \backslash W / W_K$ in W. For brevity, we will use the following notation $W_K^L := W_K \cap W^L$. Additionally, let $T = \cup_{w \in W} wSw^{-1}$ be the set of reflections of W. If $\mathcal{P}(T)$ denotes the power set of

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T, then we let $N : W \to \mathcal{P}(T)$ be given by $N(w) = \{t \in T \mid \ell(tw) < \ell(w)\}$. We define a new set $\widetilde{J} = \{wsw^{-1} \mid w \in W_I;\ s \in J\} \subset T$, and then let $\widetilde{W}$ be the subgroup of W generated by $\widetilde{J}$. Part a) of the following result is due to [4, Proposition 2.1]; part b), along with a stronger version of part a), is proven (independently) in [1].

Theorem 1.1 (Gal; Bonnafé, Dyer)**.** *Let $(\widetilde{W}, \widetilde{J})$ be as above.*

a) *$(\widetilde{W}, \widetilde{J})$ is a Coxeter system and $W = \widetilde{W} \rtimes W_I$.*
b) *Every element $w \in W_I$ is the unique element of the coset $w\widetilde{W}$ with minimal length.*

We complement [1] by noting that any normal reflection subgroup of a Coxeter group appears as the normal component of a semidirect product decomposition for W as in Theorem 1.1.

Proposition 1.2. *Let W' be any normal reflection subgroup of W. Then, there exists a subset $I \subset S$ such that $W = W_I \ltimes W'$, that is, any normal reflection subgroup of W is a normal semidirect factor of W.*

Proof. Let W'' be any reflection subgroup of W and let W' be its normal closure in W. Then, define $J \subset S$ to be $J = S \cap \left(\bigcup_{w \in W} w\,(W'' \cap T)\,w^{-1}\right)$. Since W' is normal and generated by the reflections it contains, we have that W' is generated by the W-conjugates of J. Now, we let $I = S \setminus J$, and let $\widetilde{J} = \{wsw^{-1} \mid w \in W_I;\ s \in J\}$ and $\widetilde{W}$ be the group generated by $\widetilde{J}$. By Theorem 1.1, we know that $\widetilde{W}$ is normal; since $J \subset \widetilde{W}$, we have $wJw^{-1} \subset \widetilde{W}$ for all w and so $W' \leq \widetilde{W}$. Thus, $W' = \widetilde{W}$. Also by the theorem, we know that $W = W_I \ltimes \widetilde{W}$ so $W = W_I \ltimes W'$. □

So any normal reflection subgroup arises from a decomposition of S into two disjoint subsets such that no element in one set is conjugate to an element of the other set.

Now, we can move on to our main result. For any element $x \in W$, define the set

$$x^{\perp} := \{s \in S \mid sr = rs \text{ for all } r \text{ appearing in a reduced expression for } x\},$$

which is well defined since any reduced expression for x uses the same set of simple reflections.

Now, for fixed $w \in W_I$, elements of the form wxw^{-1} with $x \in W_J$ form a standard parabolic subgroup of $\widetilde{W}$, namely the subgroup generated by the set

$$J' = \{wsw^{-1} \mid s \in J\} \subset \widetilde{J}.$$

In Lemma 2.4 we will show that every W_I-conjugate of an element in W_J is uniquely expressible as wxw^{-1} with $w \in W_I^{I \cap x^{\perp}}$ and $x \in W_J$. Our goal is to understand how reduced expressions of elements of this form reflect the semidirect product decomposition of W.

For any expression $r_1 ... r_n$ with $r_i \in S$ for all i and $D \subset \{1, ..., n\}$, we denote by $\prod_D r_i$ the product $r_{i_1} ... r_{i_j}$ where $i_1 < i_2 < \cdots < i_j$ and $D = \{i_1, ..., i_j\}$. The main result here characterizes the reduced expressions of elements of the form wxw^{-1} where $w \in W_I$ and $x \in W_J$.

Theorem 1.3. *Suppose that $S = I \cup J$, $I \cap J = \emptyset$, and no element of I is conjugate to an element of J, and define $(\widetilde{W}, \widetilde{J})$ as above. Suppose that $x \in W_J$ and $w \in W_I^{I \cap x^{\perp}}$ with $\ell(x) = m$. Fix a reduced expression $wxw^{-1} = r_1 ... r_d$ for wxw^{-1}. From* *this expression, we can define two sets* $A_J := \{i|\ r_i \in J\}$ *and* $A_I := \{i|\ r_i \in I\}$. *Then the following are true:*

(1) $|A_J| = m$.
(2) $\prod_{A_J} r_i$ *is a reduced expression of* x.
(3) *There exists* $B \subset A_I$ *where* $\prod_B r_i$ *is a reduced expression of* w *and* $\prod_{A_I \setminus B} r_i$ *is a reduced expression of* w^{-1}.
(4) *Let* $i \in A_J$, *so* $r_i \in J$. *Then*
  (a) $r_i r_j = r_j r_i$ *for all* $j \in B$ *with* $j > i$.
  (b) $r_i r_k = r_k r_i$ *for all* $k \in A_I \setminus B$ *with* $k < i$.
(5) *Let* $i \in B$ *and* $j \in A_I \setminus B$ *with* $j < i$. *Then* $r_i r_j = r_j r_i$.
(6) $N(wxw^{-1}) \cap \widetilde{W} = wN(x)w^{-1}$.

This theorem basically says that every reduced expression for wxw^{-1} is essentially of the form: (reduced expression of w)(reduced expression of x)(reduced expression of w^{-1}) up to commutation relations. Following this, we can say a bit more for the case of x being a reflection $x = t \in W_J \cap T$.

Corollary 1.4. *Consider the same setup as Theorem 1.3. Let* $w \in W_I^{I \cap x^\perp}$ *and* $x = t \in W_J \cap T$ *with* $\ell(t) = 2k+1$ *so that* $wtw^{-1} = r_1 ... r_{e+1} ... r_{2e+1}$ *is a reduced expression. Then following the notation from Theorem 1.3, the following are true:*

(1) $e + 1 \in A_J$ *and* $|\{i \in A_J|\ i < e+1\}| = k$.
(2) $B = \{i \in A_I|\ i < e+1\}$ *and so* $A_I \setminus B = \{i \in A_I|\ i > e+1\}$.

2. Proof of Theorem 1.3

To prove the main result, we need two auxiliary results. To get to these results, we first remind the reader of the following standard results regarding shortest double coset representatives.

Proposition 2.1. *Let* $K, L \subset S$. *The following hold:*

(1) *For* $w \in {}^K W^L$, $W_K \cap wW_Lw^{-1} = W_{K \cap wLw^{-1}}$.
(2) *Let* $w \in {}^K W^L$, *then any* $x \in W_K w W_L$ *has a unique expression* $x = uwv$ *with* $u \in W_K^{K \cap wLw^{-1}}$ *and* $v \in W_L$. *Furthermore, for this expression,* $\ell(uwv) = \ell(u) + \ell(w) + \ell(v)$.

Proof. See [2, 2.7]. The proofs given are for finite Coxeter groups, but the proofs easily extend to any Coxeter group. □

This proposition is the main ingredient for the proofs of Lemma 2.3 and Lemma 2.4 below. To prove these lemmas, it is also useful to have the following:

Lemma 2.2. *Let* $I, J \subset S$, $I \cap J = \emptyset$, *and let* $x \in W_J$. *Then* $x \in {}^I W^I$ *and* $I \cap xIx^{-1} = I \cap x^\perp$.

Proof. Clearly multiplying x by a simple reflection from I on the left or right must increase the length so $x \in {}^I W^I$. Now the fact that $I \cap xIx^{-1} \subset I \cap x^\perp$ follows since if $r = xsx^{-1}$ for some $r, s \in I$, then $rx = xs$. Now, both rx and xs are reduced expressions (for any reduced expression of x) and since all reduced expressions use the same simple reflections we must have $r = s$ since $x \in W_J$. Thus we have $r \in x^\perp$. Containment in the other direction is trivial. □

Lemma 2.3. *Let $I, J \subset S$ with $I \cap J = \emptyset$. Let $x \in W_J$, and let $w \in W_I^{I\cap x^\perp}$ (that is, w is in W_I and is a shortest right coset representative of $W_{I\cap x^\perp}$). Then $\ell(wxw^{-1}) = 2\ell(w) + \ell(x)$.*

Proof. This is just a combination of Proposition 2.1 and Lemma 2.2. □

The next lemma, which follows from Proposition 2.1, gives a normal form for W_I-conjugates of reflections in W_J.

Lemma 2.4. *Suppose that $S = I \cup J$ and $I \cap J = \emptyset$. Every W_I-conjugate of an element in W_J, wxw^{-1} ($w \in W_I$, $x \in W_J$) is uniquely expressible as uxu^{-1} with $u \in W_I^{I\cap x^\perp}$ and $x \in W_J$.*

Proof. Since $x \in {}^IW^I$ then by Proposition 2.1 part (2), we know that $wxw^{-1} = uxv$ with $u \in W_I^{I\cap x^\perp}$ (by Lemma 2.2) and $v \in W_I$. We note that $w = u'w'$ where $u' \in W_I^{I\cap x^\perp}$ and $w' \in W_{I\cap x^\perp}$ so that $wxw^{-1} = u'w'xw^{-1} = u'xw'w^{-1} = u'xu'^{-1}$ so that by necessity $u' = u$ and so $v = u^{-1}$ in the unique decomposition of wxw^{-1}. Thus, $wxw^{-1} = uxu^{-1}$ with $u \in W_I^{I\cap x^\perp}$. □

Remark 2.5. Lemma 2.4 can also be interpreted in the following way. Let $x, x' \in W_J$ and $w, w' \in W_I$ then $wxw^{-1} = w'x'w'^{-1}$ if and only if $x = x'$ and $w^{-1}w' \in W_{I\cap x^\perp}$.

From this point on, we will assume that $S = I \cup J$ with $I \cap J = \emptyset$ and no element of I is conjugate to an element of J.

Remark 2.6. Before we move the proof of Theorem 1.3, notice that since $\widetilde{W} \lhd W$, then for w and x as in Theorem 1.3 with $wxw^{-1} = r_1....r_d$ we have $t_i := r_1...r_{i-1}r_ir_{i-1}...r_1 \in \widetilde{W}$ if and only if $r_i \in J$.

Proof of Theorem 1.3. Suppose that $w = u_1...u_n$ and $x = s_1...s_m$ are reduced expressions for w and x respectively. Then, we know by Lemma 2.3 that $wxw^{-1} = u_1...u_ns_1...s_mu_n...u_1$ is a reduced expression for wxw^{-1}. Therefore, according to Remark 2.6, $|N(wxw^{-1}) \cap \widetilde{W}| = \ell(x)$. Using the remark in reverse, if $wxw^{-1} = r_1...r_{2n+m}$ is any reduced expression, there must be $i_1 < i_2 < \cdots < i_k$ such that $r_{i_j} \in J$ for all j and $k = \ell(x)$. Then $A_J = \{i_1, ..., i_{\ell(x)}\}$, and this proves (1).

Now, any reduced expression can be achieved by applying braid operations to a given reduced expression. We already have a particular reduced expression, namely $wxw^{-1} = u_1...u_ns_1...s_mu_n...u_1$, so $A_J = \{n + 1, ..., n + m\}$, $A_I = \{1, ..., n, n + m + 1, ..., 2n + m\}$ and $B = \{1, ..., n\}$. Thus (2)-(5) clearly hold for this reduced expression. Now we proceed to prove (2)-(5) by induction on the number of braid operations needed to change from $u_1...u_ns_1...s_mu_n...u_1$.

Suppose $wxw^{-1} = r_1...r_{2n+m}$ is a reduced expression satisfying (2)-(5) of the theorem. For this expression we have A_I, A_J and B defined. Clearly, braid operations involving simple reflections, r_i, with indices, i, that are both in A_J, both in B, or both in $A_I \setminus B$ are just rearranging x, w, or w^{-1} respectively and are thus giving new reduced expressions for x, w, or w^{-1} respectively; consequently, the theorem holds. This leaves us with three cases.

Case 1: $r_i \in I$ and $r_{i+1} \in J$ (or vice versa). If r_i and r_{i+1} do not commute, then we know that $r_{i-1} = r_{i+1} \in J$ or $r_{i+3} = r_{i+1} \in J$ by the restriction that no element of I is conjugate to an element of J. However, either of these situations contradicts the fact that $\prod_{A_J} r_i = r_{i_1}...r_{i-1}r_{i+1}r_{i+3}...r_{i_{\ell(x)}}$ is a reduced expression of x, which holds because $wxw^{-1} = r_1...r_{2n+m}$ is a reduced expression satisfying (2). Therefore, r_i commutes with r_{i+1} and thus $wxw^{-1} = r_1 ... r_{i+1} r_i ... r_{2n+m}$ is a reduced expression satisfying (2)-(5).

Case 2: $i \in B$ and $i+1 \in A_I \setminus B$ so $r_i \in I$ and $r_{i+1} \in I$. Let $r := r_i$ and $s := r_{i+1}$ and suppose that these simple reflections do not commute. Then to perform a braid operation, either $r_{i-1} = s$ or $r_{i+2} = r$ (or both). Without loss of generality, we assume $r_{i+2} = r$. We note that in this case we must have $i+2 \in A_I \setminus B$ otherwise we contradict the fact that $\prod_B r_i$ is a reduced expression for w. However, by induction, we know that $r = r_i$ commutes with all r_j with $j < i$ and $j \in A_J$ and simultaneously, $r = r_{i+2}$ commutes with all r_j with $j > i+2$ and $j \in A_J$. This implies that r commutes with all r_j with $j \in A_J$ so that r commutes with x. Finally, we see also by induction that $r = r_{i+2}$ commutes with all r_k with $k \in B$ and $k > i$. Thus, w has a reduced expression ending in r and r commutes with x contradicting the fact that $w \in W_I^{I \cap x^{\perp}}$. So, r_i must commute with r_{i+1} and $wxw^{-1} = r_1 ... r_i r_{i+1} r_{i+2} ... r_{2n+m}$ is a new reduced expression satisfying (2)-(5).

Case 3: $i \in A \setminus B$ and $i+1 \in B$ so $r_i \in I$ and $r_{i+1} \in I$. Since the reduced expression we are currently in satisfies the theorem by induction, property (5) implies that r_i and r_{i+1} commute.

So we have shown that any braid operation that can occur must either be a commutation relation or be a rearrangement of w, x, or w^{-1} separately.

Lastly, we prove part (6). As we have seen, $|N(wxw^{-1}) \cap \widetilde{W}| = \ell(x) = |N(x)|$. Clearly, $wN(x)w^{-1} \subset \widetilde{W}$ since $x \in W_J$, $W_J \subset \widetilde{W}$ and $\widetilde{W}$ is normal. So let $t \in N(x)$, that is $\ell(tx) < \ell(x)$. Since $\ell(wxw^{-1}) = 2\ell(w) + \ell(x)$, then $\ell(wtw^{-1}wxw) = \ell(wtxw^{-1}) < \ell(wxw^{-1})$ because tx has smaller length than x. Thus, $wtw^{-1} \in N(wxw^{-1})$ for all $t \in N(x)$, and so $wN(x)w^{-1} \subset N(wxw^{-1})$. Finally, we have seen that $wN(x)w^{-1} \subset N(wxw^{-1}) \cap \widetilde{W}$ and these sets are both finite with the same cardinality so they must be equal. ☐

Lastly, the corollary follows from the following information about reflections. For a reflection t, we say that $t = r_1 r_2 .... r_n$ is a palindromic reduced expression if $r_i = r_{n-i+1}$ for all $i \leq \frac{n-1}{2}$. The following lemma is due to [3].

Lemma 2.7. *If $t = r_1 ... r_{2m+1}$ is a reduced expression for $t \in T$, then $t = r_1 ... r_m r_{m+1} r_m ... r_1$ is a reduced palindromic expression of t.*

Corollary 2.8. *Suppose I and J are disjoint, $S = I \cup J$ and no element of I is conjugate to any element of J. Let $w \in W_I$ and $t \in W_J \cap T$. Now, for any reduced expression $wtw^{-1} = r_1 ... r_{n+1} ... r_{2n+1}$ then we have $wtw^{-1} = r_1 ... r_n r_{n+1} r_n ... r_1$ and so $r_{n+1} \in J$.*

Proof. This is just a restatement of Lemma 2.7 along with the fact that $wtw^{-1} \in \widetilde{W}$ so that wtw^{-1} must be conjugate to an element of J. ☐

Proof of Corollary 1.4. Let $wtw^{-1} = r_1 ... r_{e+1} ... r_{2e+1}$. The proof follows directly from Corollary 2.8, which tells us that $e+1 \in A_J$. Thus, we can never have a braid operation occur between the middle element and a simple reflection in I. Therefore, we can never have $i \in B$ and $j \in A_I \setminus B$ with $j < i$ since that would have required a braid relation between r_i and the middle element or r_j and the middle element, which would in turn have given us a reduced expression for wtw^{-1} with $r_{e+1} \in I$. ☐

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