SOME COMBINATORICS OF RATIONAL BASE REPRESENTATIONS

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Abstract. We investigate representations of natural numbers in terms of rational bases. We then define a collection of combinatorial data associated with a rational number and describe the properties of these data. We use this combinatorial data to construct an infinite, labeled, rooted tree and prove that it contains many of the interesting features of the relevant rational base representations of natural numbers. We also explore some of the combinatorics of these trees as well as the rational base representations of interest. Our main result is an analog of Kummer's theorem for rational bases representations in terms of generalized binomial coefficients defined in terms of sequences generated from the previously mentioned rooted trees.

Introduction

In [4] and [5], many connections are established between base-b representations of numbers, number theory, and some associated combinatorial objects. In an attempt to generalize those results to rational base representations of numbers, we encountered some difficulties. In rectifying the difficulties, we learned a great deal about these rational base representations and different combinatorial objects associated with them. We explore some known connections between rational base representations of natural numbers elementary number theory by providing a simple construction of a class of infinite, rooted trees. With these objects understood, we are able to generalize the main result of [4] to rational bases.

The preliminary section provides the appropriate background necessary to discuss our main result. Many of the results we include are already known; they either appear in [2], [9], [3], or [15] or are consequences of results therein. As these results are spread out over many papers, we provide proofs for completeness. We begin by introducing our chosen notion of a $\frac{p}{q}$-representation of a natural number, which can

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be found in [9]; it is well known that every natural number has a unique such representation when p and q are relatively prime. These representations differ from those described in [18] as we use a different digit set altogether; they also differ slightly from the main representations given in [2] and [17]. We also include few results pertaining to performing arithmetic with these representations and include an analogs of classical results about the sum-of-digits function and the length function. Next, we describe a collection of combinatorial data uniquely determined by a pair of relatively prime natural numbers that we call a $\frac{p}{q}$-branch sequence. We explain a method similar to the Euclidean Algorithm to compute these sequences and their associated cobranch sequences. As it turns out, these sequences are closely related to ideas that have already been extensively studied. Since our objects differ slightly, we provide proofs of results that are well known in the other contexts but may not have been applied explicitly to the sequences in the form we describe. As a side note, we explore a connection between branch sequences and a famous binary tree from number theory called the Stern-Brocot tree.

In Section 2, we employ the properties of branch sequences to construct an infinite, labeled, rooted tree in two different ways. The first construction is basically the same as the one given in [2] and [9] where it is used to describe an alternate family of rational base expansions. We then provide a simple combinatorial construction of the same tree and prove these two trees are the same. While we believed this combinatorial construction to be new, we have since been notified that our construction is equivalent to the recent construction of trees in [16]; nonetheless, we include our construction as it was discovered independently. Finally, we provide proofs of known results that this tree allows us to determine if a sequence of digits corresponds to a $\frac{p}{q}$-representation of a natural number. As a byproduct, we produce a formula for finding any given digit of a representation and characterize the numbers that are divisible by q.

Section 3 contains our main result. We discuss the motivation for this work by introducing the dominance order on $\mathbb{N}$ arising from $\frac{p}{q}$-representations. We utilize our understanding of $\frac{p}{q}$-representations and the associated trees to to provide a few results analogous to those in [4] and [5]. In particular, we introduce a family of generalized binomial coefficients (as described in [13]) and use these coefficients to prove an analog of famous theorem due to Kummer for $\frac{p}{q}$-arithmetic.

For a detailed treatment about various positional numeral systems see [12], [14], or [10].

1. Preliminaries and Background

As mentioned in the introduction, most of the results in this section can be found in, or are consequences of results from, [2], [9], [3], or [15]. However, our proofs are entirely elementary, requiring no background beyond standard undergraduate courses.

$\frac{p}{q}$-representations of natural numbers. For each $b \in \mathbb{N}$ with $b \geq 2$, we let $A_b = \{0, 1, ..., b-1\}$ and call this the *digit set for* b. Next, we let

$$S_b = \{(a_0, a_1, a_2, ...) \mid a_i \in A_b \text{ and } a_i = 0 \text{ for all but finitely many } i\}.$$

For ease of notation, we write elements of S_b as $(a_0, a_1, ..., a_k)$ where k is the largest index with $a_k \neq 0$. For $s = (a_0, a_1, \ldots, a_k) \in S_b$, we say that s has *length* k and write $\operatorname{len}(s) = k$.

Now, let $p, q \in \mathbb{N}$ with $p > q \geq 1$. We define the (p, q)-evaluation map $e_{p,q} : S_p \to \mathbb{Q}$ by

$$e_{p,q}((a_0, a_1, ..., a_k)) = \sum_{i=0}^{k} a_i \left(\frac{p}{q}\right)^i.$$

For any $n \in \mathbb{N}$, we say $(a_0, a_1, \ldots, a_k) \in S_p$ is a $\frac{p}{q}$-*representation* for n if $e_{p,q}((a_0, a_1, \ldots, a_k)) = n$, and we will denote this by $n = (a_0, a_1, \ldots a_k)_{\frac{p}{q}}$. See A024629-40 in [1] for various examples. When $q = 1$, this is the standard definition of a base-p representation of a natural number. Moreover, each natural number has a unique $\frac{p}{1}$-representation. We extend this result as follows.

Theorem 1.1. *For every $p, q \in \mathbb{N}$ with $p > q \geq 1$ and $\gcd(p, q) = 1$; each $n \in \mathbb{N}$ has a unique $\frac{p}{q}$-representation.*

Proof. The existence of these representations follows by induction and the fact that $p\left(\frac{p}{q}\right)^i = q\left(\frac{p}{q}\right)^{i+1}$.

The common proof of uniqueness is as follows. We assume, for the sake of contradiction, that n has two distinct $\frac{p}{q}$-representations $n = (a_0, a_1, \ldots, a_m)_{\frac{p}{q}}$ and $n = (b_0, b_1, \ldots, b_s)_{\frac{p}{q}}$ so that $a_i, b_i \in \{0, 1, ..., p-1\}$ for all i. Assume $m \leq s$, and let j be the minimal index where $a_j \neq b_j$. Then, it follows that

$$0 = \sum_{i=0}^{m} a_i \left(\frac{p}{q}\right)^i - \sum_{i=0}^{s} b_i \left(\frac{p}{q}\right)^i.$$

Now, since $a_i = b_i$ for all $i < j$ we get

$$0 = (a_j - b_j)\left(\frac{p}{q}\right)^j + \sum_{i=j+1}^{s} (a_i - b_i)\left(\frac{p}{q}\right)^i.$$

Dividing each term by $\left(\frac{p}{q}\right)^j$ and multiplying by q^{s-j} leaves us with

$$0 = q^{s-j}(a_j - b_j) + \sum_{i=j+1}^{s} (a_j - b_j)(p)^{i-j} q^{s-i}.$$

Since $s \geq i > j$ for all i, we know that this is an integral equation. Therefore $0 = q^{s-j}(a_j - b_j) \pmod{p}$. Now, the fact that $\gcd(p, q) = 1$ implies that

$$0 = a_j - b_j \pmod{p},$$

and so $a_j = b_j \pmod{p}$; however, since $a_j, b_j \in \{0, 1, ..., p-1\}$, this means $a_j = b_j$, which is a contradiction. $\square$

Example 1.2. Let $p = 3$ and $q = 2$. Then the following table lists the $\frac{3}{2}$-representations of the integers from 0 to 21 (we have omitted subscripts for aesthetics). See A024629 in [1] for more data.

n	$\frac{3}{2}$-representation	n	$\frac{3}{2}$-representation
0	(0)	11	(2,0,1,2)
1	(1)	12	(0,2,1,2)
2	(2)	13	(1,2,1,2)
3	(0,2)	14	(2,2,1,2)
4	(1,2)	15	(0,1,0,1,2)
5	(2,2)	16	(1,1,0,1,2)
6	(0,1,2)	17	(2,1,0,1,2)
7	(1,1,2)	18	(0,0,2,1,2)
8	(2,1,2)	19	(1,0,2,1,2)
9	(0,0,1,2)	20	(2,0,2,1,2)
10	(1,0,1,2)	21	(0,2,2,1,2)

For example, $1(\frac{3}{2})^0 + 0(\frac{3}{2})^1 + 1(\frac{3}{2})^2 + 2(\frac{3}{2})^3 = 10$ and so $10 = (1, 0, 1, 2)_{\frac{3}{2}}$.

Remark. The uniqueness of Theorem 1.1 requires p and q to be relatively prime. For example $25 = (5,4)_{\frac{10}{2}}$ and $25 = (0,5)_{\frac{10}{2}}$. Consequently, for the rest of the paper, we will only consider the cases where p and q are relatively prime.

Of course, the $\frac{p}{q}$-representation of $n+m$ can be generated from the representations of n and m separately. More precisely, suppose that $n, m \in \mathbb{N}$ with $n = (n_0, n_1, \ldots, n_k)_{\frac{p}{q}}$ and $m = (m_0, m_1 \ldots, m_s)_{\frac{p}{q}}$. We first define the $(\frac{p}{q}, n, m)$-*carry sequence* recursively by

$$\epsilon_i^{\frac{p}{q},n,m} = \left\lfloor \frac{n_i + m_i + q\epsilon_{i-1}}{p} \right\rfloor,$$

where we set $\epsilon_{-1}^{\frac{p}{q},n,m} = 0$. When the context is understood, we will omit the superscripts. Using this carry sequence, the $\frac{p}{q}$-representation of $n+m$ is given by

$$(n+m)_i = n_i + m_i + q\epsilon_{i-1} - p\epsilon_i. \tag{1}$$

The formula listed in Equation (1) provides us with a system of base-$\frac{p}{q}$ arithmetic. We mention two interesting features of this arithmetic system.

Proposition 1.3. *Let $n, m, p, q \in \mathbb{N}$ with $p > q \geq 1$ and $\gcd(p,q) = 1$. For all $i \in \mathbb{N}$*

$$\epsilon_i^{\frac{p}{q},n,m} \leq q.$$

Proof. By definition $\epsilon_{-1} = 0 \leq q$. Assume $\epsilon_{i-1} \leq q$. Then

$$\frac{n_i + k_i + q\epsilon_{i-1}}{p} \leq \frac{2(p-1) + q^2}{p} = \frac{2p + q^2 - 2}{p} < \frac{2p + pq - p}{p} = q + 1,$$

where the last inequality holds since $q^2 - 2 < pq - p$ for all $p > q \geq 1$. Since the inequality is strict, we see that

$$\epsilon_i = \left\lfloor \frac{n_i + m_i + q\epsilon_{i-1}}{p} \right\rfloor \leq q.$$

The result follows by induction. □

Thus there is a bound on the size of carries we must keep track of while adding two numbers. This proposition is a consequence of [2] that addition in this arithmetic can be realized by a finite state automaton.

The next proposition provides a formula for determining the total number of carries when adding two numbers in base-$\frac{p}{q}$ arithmetic. To do this we define the $\frac{p}{q}$-sum-of-digits function by $\mathrm{sum}_{\frac{p}{q}}(n) = \sum_{i=0}^{k} n_i$

when $n = (n_0, n_1, \dots, n_k)_{\frac{p}{q}}$. For examples of these functions, see A244040-41, A245336-39 in [1].

Proposition 1.4. *Let $p, q \in \mathbb{N}$ with $p > q$ and $\gcd(p, q) = 1$. For all $n, m \in \mathbb{N}$*

$$\operatorname{sum}_{\frac{p}{q}}(n) + \operatorname{sum}_{\frac{p}{q}}(m) - \operatorname{sum}_{\frac{p}{q}}(n+m) = (p-q)\sum_{i=0}^{t} \epsilon_i$$

where t is the largest index such that $(n+m)_t$ is nonzero.

Proof. We assume $n = (n_0, n_1, \dots, n_s)_{\frac{p}{q}}$, $m = (m_0, m_1, \dots, m_u)$ and $n + m = ((n+m)_0, (n+m)_1, \dots, (n+m)_t)$ where $n_i, m_i, (n+m)_i \in \{0, 1, ..., p-1\}$ for all i and $t \geq u \geq s$. Then substituting Equation (1) into $\operatorname{sum}_{\frac{p}{q}}(n) + \operatorname{sum}_{\frac{p}{q}}(m) - \operatorname{sum}_{\frac{p}{q}}(n+m)$ yields

$$\sum_{i=0}^{s} n_i + \sum_{i=0}^{u} m_i - \sum_{i=0}^{t}(n_i + m_i + q\epsilon_{i-1} - p\epsilon_i).$$

After canceling appropriate terms and reorganizing, this is equal to

$$-q(\epsilon_{-1} + \epsilon_0 + \cdots + \epsilon_{t-1}) + p(\epsilon_0 + \epsilon_1 + \cdots + \epsilon_t).$$

Since $\epsilon_{-1} = \epsilon_t = 0$, we conclude

$$\operatorname{sum}_{\frac{p}{q}}(n) + \operatorname{sum}_{\frac{p}{q}}(m) - \operatorname{sum}_{\frac{p}{q}}(n+m) = (p-q)(\epsilon_0 + \cdots + \epsilon_{t-1} + \epsilon_t)$$

as required. $\square$

When $q = 1$, this is a classical result about the sum of digits that demonstrates the interconnected nature of number theory, algebraic combinatorics, and enumerative combinatorics (see [4]).

Since every natural number has a unique $\frac{p}{q}$-representation, we let $\operatorname{len}_{\frac{p}{q}}(n) = k$ when $n = (n_0, n_1, \dots, n_k)_{\frac{p}{q}}$. Proposition 1.3 provides a bound on the length of $n + m$ in terms of n and m, implies that $0 \leq \operatorname{len}_{\frac{p}{q}}(n) - \operatorname{len}_{\frac{p}{q}}(n-1) \leq 1$. We are particularly interested in the case where $\operatorname{len}_{\frac{p}{q}}(n) - \operatorname{len}_{\frac{p}{q}}(n-1) = 1$; when n has this property we refer to n as a *knockout carry* (because when $n-1$ and 1 are added, each digit is reduced by q, except for the first digit which is reduced by $p-1$ to 0). To describe the knockout carries, we let $g_0 : \mathbb{N} \to \mathbb{N}$ be the function

$$g_0(n) = p\left\lceil \frac{n}{q} \right\rceil.$$

This function will also be important in Section 2.

Based on results in [2] or by direct computation we see that if n is a knockout carry then $g_0(n)$ is the smallest knockout carry greater than n. The following proposition is then immediate.

Proposition 1.5. *The sequence of knockout carries* $(k_1, k_2, k_3, \ldots)$ *is determined by* $k_1 = 1$ *and* $k_i = g_0(k_{i-1})$ *for all* $i > 1$. *Moreover, if* c_n *is the number of nonnegative integers with length* n, *then for* $n > 1$, $c_n = k_{n+1} - k_n$.

For an example of these two different types of sequences when $p = 3$ and $q = 2$, see A070885 and A081848 in [1].

In [5] the authors describe in detail a partial order defined in terms of base-b representations where $b \in \mathbb{N}$. In this special case, every sequence in S_b corresponds to a natural number under the $(b, 1)$-evaluation map. Unfortunately, this is not true for $\frac{p}{q}$-representations in general. As such, the associated partial order, defined in Section 3, is more complicated to understand because it requires us to know which sequences in S_p correspond to natural numbers. In [2] and [9], the authors define an infinite, rooted tree that characterizes their corresponding rational base representations. In the following subsections, we introduce a family of combinatorial data that we will use to give a purely combinatorial construction of the same tree in Section 2.

$\frac{p}{q}$-branch sequences. For this section, we fix two natural numbers, p and q, satisfying $p > q \geq 1$ and $\gcd(p, q) = 1$. We begin by introducing a collection of combinatorial data that we call a $\frac{p}{q}$-branch sequence. It turns out that $\frac{p}{q}$-branch sequences are closely related to well-studied notions such as Euclidean strings, Christoffel words, and Sturmian sequences (see [8], [6], [7]). Furthermore, after completion of this project we learned that much of this information has been given independently in [16]. Nonetheless, we include our construction and (re)prove some properties that are known about those objects for completeness.

We let $\mathbf{t}$ be the sequence $\mathbf{t} = (t_1, \ldots, t_{q+1})$ given by $t_i = \left\lceil \frac{ip}{q} \right\rceil$ ($\mathbf{t}$ is actually an infinite sequence that we truncate and is sometimes referred to as a Beatty sequence, see A007494 in [1] for instance). Then, we define the $\frac{p}{q}$-*branch sequence* to be $\mathbf{b} = (b_1, \ldots b_q)$ where $b_i = t_{i+1} - t_i$.

We will use bold letters to represent branch sequences in order to differentiate them from $\frac{p}{q}$-representations. Technically, the sequence $\mathbf{b}$ just defined can be thought of as an infinite, periodic sequence with period q; we may often refer to b_k when $k > q$, in which case we mean $b_{(k \bmod q)}$. For example, see A040001 and A107453 in [1] for the $\frac{3}{2}$- and $\frac{5}{4}$-branch sequences respectively. Each number t_i counts the multiples of q strictly less than ip including 0. Consequently, b_i counts the multiples of q between ip and $(i+1)p$ including ip if $ip = 0 \pmod q$. Moreover, the following properties are satisfied by the $\frac{p}{q}$-branch sequence.

Proposition 1.6. *Let* $\mathbf{b} = (b_1, ..., b_q)$ *be the* $\frac{p}{q}$*-branch sequence. Then the following hold.*

(1) *If* $q = 1$, $\mathbf{b} = (p)$,
(2) *the sum of the entries is* p, *i.e.* $\sum_{i=1}^{q} b_i = p$,
(3) *for all* i, $\left\lfloor \frac{p}{q} \right\rfloor \leq b_i \leq \left\lceil \frac{p}{q} \right\rceil$,
(4) *the penultimate entry* $b_{q-1} = \left\lfloor \frac{p}{q} \right\rfloor$, *and*
(5) *the final entry* $b_q = \left\lceil \frac{p}{q} \right\rceil$.

Proof. Part (1) follows directly from the definition. For part (2), we have $\sum_{i=1}^{q} b_q = t_{q+1} - t_1 = \left\lceil \frac{(q+1)p}{q} \right\rceil - \left\lceil \frac{p}{q} \right\rceil = p + \left\lceil \frac{p}{q} \right\rceil - \left\lceil \frac{p}{q} \right\rceil = p$. Now, for part (3), we note that $1 - \{x\} = \lceil x \rceil - x$ where $\{x\}$ represents the fractional part of x. Using this result, we get

$$b_i = \left\lceil \frac{(i+1)p}{q} \right\rceil - \left\lceil \frac{ip}{q} \right\rceil = \left\lceil \frac{ip}{q} \right\rceil + \left\lceil \frac{p}{q} \right\rceil + \left\lceil \left\{ \frac{ip}{q} \right\} + \left\{ \frac{p}{q} \right\} \right\rceil - 2 - \left\lceil \frac{ip}{q} \right\rceil.$$

Then, since $1 \leq \left\lceil \left\{ \frac{ip}{q} \right\} + \left\{ \frac{p}{q} \right\} \right\rceil \leq 2$, we have $\left\lceil \frac{p}{q} \right\rceil - 1 \leq b_i \leq \left\lceil \frac{p}{q} \right\rceil$, as required. For parts (4) and (5) we see that

$$b_{q-1} = \left\lceil \frac{qp}{q} \right\rceil - \left\lceil \frac{(q-1)p}{q} \right\rceil = - \left\lceil -\frac{p}{q} \right\rceil = \left\lfloor \frac{p}{q} \right\rfloor,$$

and

$$b_q = \left\lceil \frac{(q+1)p}{q} \right\rceil - \left\lceil \frac{qp}{q} \right\rceil = \left\lceil \frac{p}{q} \right\rceil,$$

where both follow from the fact that $\lceil n+x \rceil = n + \lceil x \rceil$ when $n \in \mathbb{N}$. $\square$

We can also determine the branch sequence in a purely number theoretic manner using a computationally simple process similar to the Euclidean Algorithm. We describe this process in the following lemma.

Lemma 1.7. *The* $\frac{p}{q}$*-branch sequence* $\mathbf{b} = (b_1, ..., b_q)$ *is the unique finite sequence of numbers satisfying the following system of equations:*

$$\begin{aligned} p &= b_q q - r_1 \\ p - r_1 &= b_1 q - r_2 \\ &\vdots \\ p - r_q &= b_q q - r_{q+1} \end{aligned}$$

where $0 \leq r_i < q$ *for all* i. *Moreover,* $r_i \neq r_j$ *whenever* $1 \leq i < j \leq q$.

Proof. First, by Proposition 1.6, $b_q = \left\lceil \frac{p}{q} \right\rceil = t_1$. Now, for each $0 \leq i \leq q+1$, there exists a unique r_i satisfying

$$ip = q \left\lceil \frac{ip}{q} \right\rceil - r_i \tag{2}$$

with $0 \leq r_i < q$. Therefore, by definition, $ip = qt_i - r_i$. Since $t_1 = b_q$ we get $p = qt_1 - r_1 = qb_q - r_1$. Now, for $1 \leq j \leq q$ we have

$$p = (j+1)p - jp = (qt_{j+1} - r_{j+1}) - (qt_j - r_j),$$

and rearranging this gives $p - r_j = q(t_{j+1} - t_j) - r_{j+1}$. Since $j \geq 1$,we know $t_{j+1} - t_j = b_j$ and so

$$p - r_j = qb_j - r_{j+1}.$$

Finally, from this construction, we deduce that $r_i = -ip \pmod{q}$. Since $\gcd(p, q) = 1$, we conclude $r_i \neq r_j$ for $1 \leq i < j \leq q$. □

The secondary sequence $\mathbf{r} = (r_1, \ldots, r_q)$, constructed in Lemma 1.7, is uniquely determined by p and q and will play a role in our investigation of $\frac{p}{q}$-representations; we call this sequence the $\frac{p}{q}$-*cobranch sequence.*

Corollary 1.8. *Let $\mathbf{b} = (b_1, \ldots, b_q)$ be the $\frac{p}{q}$-branch sequence and $\mathbf{r} = (r_1, \ldots, r_q)$ be the corresponding cobranch sequence.*

(1) *For each i, $r_i + kq < p$ for all $k < b_i$.*
(2) *For each i, $r_i + qb_i \geq p$.*
(3) *For each i, $r_i = -ip \pmod{q}$.*
(4) *If $v = jp$ for some j, then $\left\lceil \frac{v}{q} \right\rceil = \frac{v+r_i}{q}$ where $i = j \pmod{q}$.*

Proof. According to the previous proof we know that

$$p - r_i = qb_i - r_{i+1} > qb_i - q$$

since $r_{i+1} < q$. Now, since $qb_i - q = q(b_i - 1)$ we have $p > q(b_i - 1) + r_i$, which implies $p > qk + r_i$ for $k < b_i$.

For part (2), we note that $0 \leq r_{i+1}$ and so $p - r_i = qb_i - r_{i+1} \leq qb_i$ and the result follows.

Part (3) is immediate from the last remark in the proof of Lemma 1.7. Finally, the previous lemma implies $r_q = 0$ and so $r_i = r_j$ whenever $i = j \pmod{q}$; thus, part (4) follows from Equation (2). □

Corollary 1.9. *Let $w \in \mathbb{N}$ and $a \in \{1, \ldots, w-1\}$ with $\gcd(a, w) = 1$. Then, for all $n \in \mathbb{N}$, if $\mathbf{c} = (c_1, ..., c_w)$ is the $\frac{wn+a}{w}$-branch sequence, then the $\frac{w(n+1)+a}{w}$-branch sequence is $(c_1 + 1, \ldots, c_w + 1)$.*

Proof. We know that $wn + a - r_i = wc_i - r_{i+1}$ for all i, and adding w to both sides yields $w(n+1) + a - r_i = w(c_i + 1) - r_{i+1}$. The result follows by Lemma 1.7. $\square$

Next, we define two functions on finite sequences. For any finite sequence $\mathbf{s}$ given by $\mathbf{s} = (s_1, s_2, ..., s_{t-2}, s_{t-1}, s_t)$, we let

$$\begin{aligned} \tau(\mathbf{s}) &= (s_1, s_2, ..., s_{t-2}, s_t, s_{t-1}), \text{ and} \\ \rho(\mathbf{s}) &= (s_2, s_3, ..., s_{t-1}, s_t, s_1). \end{aligned} \tag{3}$$

Thus, τ swaps the last two entries of $\mathbf{s}$ and ρ rotates the first entry of $\mathbf{s}$ to the last. In the special case where $\mathbf{s} = (s_1)$, we let $\tau(\mathbf{s}) = \mathbf{s}$ and $\rho(\mathbf{s}) = \mathbf{s}$. We write $\rho^k(\mathbf{s})$ to denote k successive applications of ρ.

Proposition 1.10. *If* $\mathbf{b}$ *is the* $\frac{p}{q}$*-branch sequence, then*

$$\tau(\mathbf{b}) = \rho^d(\mathbf{b})$$

where d is the smallest positive integer satisfying $dp = 1 \pmod q$.

Proof. When $q = 1$, the statement is clear by convention, so we may assume $q > 1$ and thus $\frac{p}{q} \notin \mathbb{N}$. Let $\mathbf{b} = (b_1, ..., b_{q-1}, b_q)$. We must show that $b_i = b_{i+d \pmod q}$ for all $1 \leq i \leq q-2$, $b_q = b_{d-1}$ and $b_{q-1} = b_d$. Recall that $b_{i+d} = b_{i+d \pmod q}$.

First, let i be fixed with $1 \leq i \leq q-2$. Then, since we know that $dp = 1 \pmod q$, it follows that $dp = 1 + qm$ for some $m \in \mathbb{Z}$. Thus,

$$\begin{aligned} b_{i+d} &= \left\lceil \frac{(i+d+1)p}{q} \right\rceil - \left\lceil \frac{(i+d)p}{q} \right\rceil \\ &= \left\lceil \frac{(i+1)p}{q} + \frac{dp}{q} \right\rceil - \left\lceil \frac{ip}{q} + \frac{dp}{q} \right\rceil \\ &= \left\lceil \frac{(i+1)p}{q} + \frac{1}{q} + \frac{mq}{q} \right\rceil - \left\lceil \frac{ip}{q} + \frac{1}{q} + \frac{mq}{q} \right\rceil \\ &= \left\lceil \frac{(i+1)p}{q} + \frac{1}{q} + m \right\rceil - \left\lceil \frac{ip}{q} + \frac{1}{q} + m \right\rceil \\ &= \left\lceil \frac{(i+1)p}{q} + \frac{1}{q} \right\rceil - \left\lceil \frac{ip}{q} + \frac{1}{q} \right\rceil. \end{aligned}$$

Now, suppose that $\left\lceil \frac{(i+1)p}{q} + \frac{1}{q} \right\rceil \neq \left\lceil \frac{(i+1)p}{q} \right\rceil$ and let $z = \left\lceil \frac{(i+1)p}{q} \right\rceil$. This assumption implies that $(i+1)p = qz - u_1$ and $(i+1)p + 1 = q(z+1) - u_2$ where $0 \leq u_1, u_2 < q$. Subtracting these two equations gives $1 = q + u_1 - u_2$, which means $u_2 = u_1 + q - 1$. However, since $u_2 \leq q - 1$, we see that $u_1 = 0$, implying $(i+1)p = qz$. Since $\gcd(p, q) = 1$, we must have $p = z$ and $q = i + 1$. Thus $i = q - 1$, which is a contradiction;

hence, $\left\lceil \frac{(i+1)p}{q} + \frac{1}{q} \right\rceil = \left\lceil \frac{(i+1)p}{q} \right\rceil$. A similar argument shows that since $i \neq q$, we have $\left\lceil \frac{ip}{q} + \frac{1}{q} \right\rceil = \left\lceil \frac{ip}{q} \right\rceil$. Thus, we see that

$$b_{i+d} = \left\lceil \frac{(i+1)p}{q} \right\rceil - \left\lceil \frac{ip}{q} \right\rceil = b_i,$$

as required.

Next, we have

$$\begin{aligned} b_d &= \left\lceil \frac{(d+1)p}{q} \right\rceil - \left\lceil \frac{dp}{q} \right\rceil \\ &= \left\lceil \frac{1+qm+p}{q} \right\rceil - \left\lceil \frac{1+qm}{q} \right\rceil = \left\lceil \frac{p+1}{q} \right\rceil - \left\lceil \frac{1}{q} \right\rceil . \end{aligned}$$

Now, $\frac{p}{q} \notin \mathbb{N}$ so $p = q\left\lceil \frac{p}{q} \right\rceil - r$ where $1 \leq r \leq q-1$. Hence $p+1 = q\left\lceil \frac{p}{q} \right\rceil - (r-1)$ where $0 \leq r-1 \leq q-1$ and thus $\left\lceil \frac{p}{q} \right\rceil = \left\lceil \frac{p+1}{q} \right\rceil$. Moreover $\left\lceil \frac{1}{q} \right\rceil = 1$; so $b_d = \left\lceil \frac{p}{q} \right\rceil - 1 = \left\lfloor \frac{p}{q} \right\rfloor = b_{q-1}$ by Proposition 1.6.

Similarly, we check that

$$b_{d-1} = \left\lceil \frac{dp}{q} \right\rceil - \left\lceil \frac{(d-1)p}{q} \right\rceil = \left\lceil \frac{1}{q} \right\rceil - \left\lceil -\frac{p}{q} \right\rceil = \left\lceil \frac{1}{q} \right\rceil + \left\lfloor \frac{p}{q} \right\rfloor .$$

Then $\left\lceil \frac{1}{q} \right\rceil = 1$, and since $\frac{p}{q} \notin \mathbb{N}$ we get $b_{d-1} = 1 + \left\lfloor \frac{p}{q} \right\rfloor = \left\lceil \frac{p}{q} \right\rceil = b_q$. $\square$

We call the property described in Proposition 1.10 the *rotate-swap* property. Euclidean strings, defined in [8], satisfy a similar property.

Proposition 1.11. *The $\frac{p}{q}$-branch sequence* $\mathbf{b}$ *is the unique length q sequence, with entries adding to p, satisfying the rotate-swap property.*

Proof. Proposition 1.10 implies that $\rho^{-1}(\tau(\mathbf{b}))$ is a Euclidean string, and Euclidean strings are shown to be unique in Lemma 1 of [8]. $\square$

We use this uniqueness to construct $\frac{p}{q}$-branch sequences using a combinatorial and number theoretic object known as the Stern-Brocot tree, which we will describe below. The following result appears to be analogous to a result proved about binary Euclidean strings (using only the digits 0 and 1) (see [8]) as well as a result about Christoffel words (see [6]).

We define the *mediant* of two fractions $\frac{a}{b}$ and $\frac{c}{d}$ to be the fraction $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$; this is often referred to as *Farey arithmetic.* We follow [11] and repeatedly apply the mediant operation to construct the Stern-Brocot tree recursively. We start with the sequence of pseudo-fractions

$f_0 = (\frac{0}{1}, \frac{1}{0})$. Then, for $i > 0$, we let f_i be the sequence obtained from f_{i-1} by inserting the mediant of two consecutive entries of f_{i-1} between those entries (we call these *mediant sequences*). For example, the first few mediant sequences are

$$f_0 = \left(\frac{0}{1}, \frac{1}{0}\right)$$
$$f_1 = \left(\frac{0}{1}, \frac{1}{1}, \frac{1}{0}\right)$$
$$f_2 = \left(\frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \frac{2}{1}, \frac{1}{0}\right)$$
$$f_3 = \left(\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \frac{3}{2}, \frac{2}{1}, \frac{3}{1}, \frac{1}{0}\right).$$

For each new mediant, $\frac{m}{n}$, introduced in the sequence f_i, we call the entries before and after $\frac{m}{n}$ in f_i the *Farey ancestors* of $\frac{m}{n}$. It is well known that if $\frac{a}{b}$ and $\frac{c}{d}$ are the Farey ancestors of $\frac{m}{n}$, then $an - bm = \pm 1$ and $cn - bm = \pm 1$ and have different parity.

Using the mediant sequences, we define the Stern-Brocot tree as follows (see also A007305 and A047679 in [1]). For each $i \geq 1$, we let the set of vertices at level i be the mediants appearing for the first time in the sequence f_i. Additionally, for each fraction $\frac{a}{b}$ at level $i \geq 2$, there is an edge to its unique Farey ancestor in level $i-1$; we call this ancestor simply the *parent* of $\frac{a}{b}$ and refer to the other Farey ancestor as the *Farey parent*. In Figure 1, we have included a picture of the Stern-Brocot tree up to level five. For instance, we see that $\frac{4}{5}$ appears for the first time in f_5, has an edge to its parent $\frac{3}{4}$ in level four, and has Farey parent equal to $\frac{1}{1}$.

As mentioned above, branch sequences are related to Christoffel words and Euclidean strings, which can both be generated using the Stern-Brocot tree. Thus, we expect the same to be true of branch sequences. To describe this connection, we note that if $\mathbf{s} = (s_1, \ldots, s_k)$ and $\mathbf{u} = (u_1, \ldots, u_l)$ are two sequences then we write $\mathbf{su}$ to mean the concatenation of the two sequences, i.e. $\mathbf{su} = (s_1, \ldots, s_k, u_1, \ldots, u_l)$.

Lemma 1.12. *Let* $\mathbf{s}$ *be any finite sequence. If* $\mathbf{s} = \mathbf{s}_1\mathbf{s}_2$ *where the length of* $\mathbf{s}_2$ *is greater than 1, we have* $\tau(\mathbf{s}) = \mathbf{s}_1\tau(\mathbf{s}_2)$.

Proof. This is clear as τ only interchanges the last two elements. $\square$

Lemma 1.13. *Let* $\frac{r}{s} > 1$ *be a reduced fraction with* $s \geq 2$*, and assume* $\frac{p}{1}$ *is a Farey ancestor of* $\frac{r}{s}$ *with* $\frac{p}{1} > \frac{r}{s}$*. Then the* $\frac{r}{s}$*-branch sequence is given by* $\mathbf{c} = (p, p, ..., p, p-1, p)$.

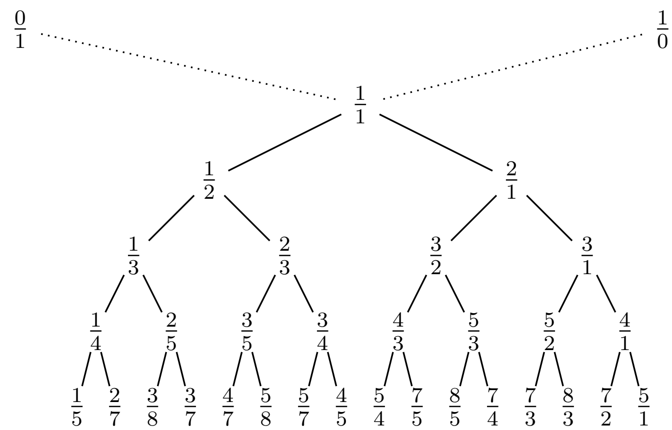

FIGURE 1. The Stern-Brocot tree.

Proof. The assumptions of the lemma imply $ps-r=1$. Thus $r=ps-1$, and the result follows by applying Lemma 1.7 to $\frac{r}{s}=\frac{ps-1}{s}$. □

Theorem 1.14. *Suppose that $\frac{v}{w}$ is a fraction satisfying $v>w\geq 2$ and $\gcd(v,w)=1$. Let $\frac{p}{q}>\frac{r}{s}$ be the Farey ancestors of $\frac{v}{w}$. If $\mathbf{b}$ is the $\frac{p}{q}$-branch sequence and $\mathbf{c}$ is the $\frac{r}{s}$-branch sequence, the $\frac{v}{w}$-branch sequence is $\tau(\mathbf{c})\mathbf{b}$.*

Proof. By assumption we know $v=p+r$ and $w=q+s$; moreover, properties of the Stern-Brocot tree guarantee $ps-rq=1$.

Now, assume the desired property holds for all ancestors of $\frac{v}{w}$, in particular for $\frac{p}{q}$ and $\frac{r}{s}$. Since $ps-qr=1$, we conclude $ps=1 \pmod q$ and $(-q)r=1 \pmod s$. So $(p+r)s-(q+s)r=ps+rs-qr-rs=pr-qs=1$, which implies $(p+r)s=1 \pmod{q+s}$. Now, let $\mathbf{d}$ be the $\frac{v}{w}$-branch sequence. We know $\rho^s(\mathbf{d})=\tau(\mathbf{d})$ since $sv=1 \pmod w$. By Proposition 1.11, $\mathbf{d}$ is the unique length w sequence whose entries sum to v with this property. Let $\mathbf{a}$ be the sequence $\mathbf{a}=\tau(\mathbf{c})\mathbf{b}$. If both $s=1$ and $q=1$, then $\mathbf{a}=\mathbf{d}$ according to the convention after (3). Otherwise $q\neq s$.

Case 1: Assume $q>s\geq 1$. This implies that $q\geq 2$, $\frac{r}{s}$ is a Farey ancestor of $\frac{p}{q}$, and $\frac{p}{q}$ is the parent of $\frac{v}{w}$. By induction $\mathbf{b}=\tau(\mathbf{c})\mathbf{b}'$ where $\mathbf{b}'$ is the sequence associated with the other Farey ancestor of $\frac{p}{q}$. By Proposition 1.10, we know that $\tau(\mathbf{b})=\rho^s(\mathbf{b})$. Notice that $\tau(\mathbf{a})=\tau(\tau(\mathbf{c})\mathbf{b})=\tau(\mathbf{c})\tau(\mathbf{b})$ by Lemma 1.12. Next, $\tau(\mathbf{c})\tau(\mathbf{b})=\tau(\mathbf{c})\rho^s(\mathbf{b})=\tau(\mathbf{c})\rho^s(\tau(\mathbf{c})\mathbf{b}')=\tau(\mathbf{c})\mathbf{b}'\tau(\mathbf{c})$ where the last equality follows since the $\tau(\mathbf{c})$ has length s. Finally, $\tau(\mathbf{c})\mathbf{b}'\tau(\mathbf{c})=\mathbf{b}\tau(\mathbf{c})=\rho^s(\tau(\mathbf{c})\mathbf{b})=\rho^s(\mathbf{a})$, and so $\tau(\mathbf{a})=\rho^s(\mathbf{a})$.

Case 2: Assume $s>q\geq 1$.

Case A: Assume $q = 1$. Then $\frac{p}{1}$ is a Farey ancestor of $\frac{r}{s}$. By Lemma 1.13, we know that $\mathbf{c} = (p, p, ..., p, p-1, p)$, which implies $\mathbf{a} = \tau(\mathbf{c})\mathbf{b} = (p, p, ..., p, p-1, p)$. In this case $s = w - 1$ and so $\rho^s(\mathbf{a}) = \rho^{w-1}(\mathbf{a}) = (p, p, ..., p, p-1)$, and $\tau(\mathbf{a}) = (p, p, ..., p, p-1)$. Hence $\rho^s(\mathbf{a}) = \tau(\mathbf{a})$.

Case B: Assume $q > 1$. This implies that $s > q \geq 2$, $\frac{p}{q}$ is a Farey ancestor of $\frac{r}{s}$, and $\frac{r}{s}$ is the parent of $\frac{v}{w}$. By induction we know that $\mathbf{c} = \tau(\mathbf{c}')\mathbf{b}$ where $\mathbf{c}'$ is the sequence associated with the other Farey ancestor of $\frac{r}{s}$. By Proposition 1.10 we know $\tau(\mathbf{c}) = \rho^{-q}(\mathbf{c}) = \rho^{s-q}(\mathbf{c})$ since $s > q$ and $s - q = -q \pmod{s}$. Now, we notice that $\tau(\mathbf{a}) = \tau(\mathbf{c})\tau(\mathbf{b})$ by Lemma 1.12. Then we have $\tau(\mathbf{c})\tau(\mathbf{b}) = \rho^{s-q}(\mathbf{c})\tau(\mathbf{b}) = \rho^{s-q}(\tau(\mathbf{c}')\mathbf{b})\tau(\mathbf{b}) = \mathbf{b}\tau(\mathbf{c}')\tau(\mathbf{b})$ since the length of $\tau(\mathbf{c}')$ is $s - q$. Finally $\mathbf{b}\tau(\mathbf{c}')\tau(\mathbf{b}) = \mathbf{b}\tau(\tau(\mathbf{c}')\mathbf{b}) = \mathbf{b}\tau(\mathbf{c}) = \rho^s(\tau(\mathbf{c})\mathbf{b}) = \rho^s(\mathbf{a})$ using Lemma 1.12 and the fact that the length of $\tau(\mathbf{c})$ is s.

In each case, we see that $\tau(\mathbf{a}) = \rho^s(\mathbf{a})$. Also, $\mathbf{a}$ has length w with entries adding to v by construction. As noted, $\mathbf{d}$ is the unique sequence with these properties, and therefore $\mathbf{d} = \mathbf{a} = \tau(\mathbf{c})\mathbf{b}$. □

Theorem 1.14 provides a method of recursively generating branch sequences for each rational number greater than 1 using the right half of the Stern-Brocot tree. First, by Proposition 1.6, we see that each integer b, which is along the right side of the right half of the Stern-Brocot tree, has $\frac{b}{1}$-branch sequence $\mathbf{b} = (b)$. Now, we construct the branch sequence for a fraction by concatenating the branch sequences of its Farey ancestors after swapping the last two entries of the left (smallest) Farey ancestor. In Figure 2, we show the first five levels of the branch tree and the corresponding branch sequences. In this example, we compare the tree to the Stern-Brocot tree in Figure 1 to see that the $\frac{8}{5}$-branch sequence is $(2, 1, 2, 1, 2)$; this is obtained by swapping the (last) two entries of the $\frac{3}{2}$-branch sequence $(1, 2)$ and concatenating with the $\frac{5}{3}$-branch sequence $(2, 1, 2)$.

We will use branch sequences to describe a combinatorial model for $\frac{p}{q}$-representations of natural numbers. The previous connection to the Stern-Brocot tree suggests that there may be an intricate relation between the $\frac{v}{w}$-representations, $\frac{p}{q}$-representations, and $\frac{r}{s}$-representations of natural numbers when $\frac{p}{q}$ and $\frac{r}{s}$ are the Farey ancestors of $\frac{v}{w}$. This seems to be the first suggestion that the Stern-Brocot tree is connected to rational base arithmetic.

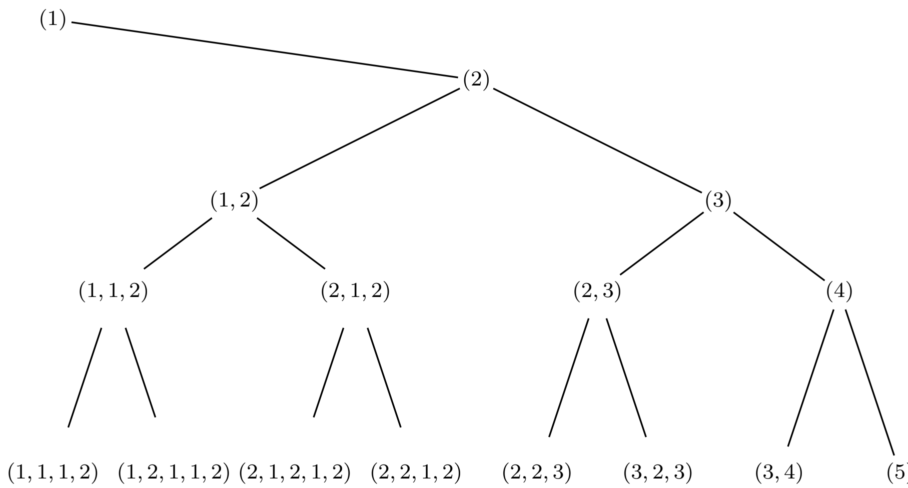

FIGURE 2. The branch sequence tree.

2. A COMBINATORIAL MODEL FOR $\frac{p}{q}$-REPRESENTATIONS

We again fix two natural numbers p and q satisfying $p > q \geq 1$ and $\gcd(p, q) = 1$. For any natural number $n \in \mathbb{N}$, we say n is of q-*type* s (or simply *type* s since q is fixed) if s is the unique integer in $\{1, \dots, q\}$ satisfying $n = s \pmod{q}$. Finally, we let $\mathbf{b} = (b_1, \dots, b_q)$ be the $\frac{p}{q}$-branch sequence and $\mathbf{r} = (r_1, \dots, r_q)$ be the corresponding cobranch sequence.

Let $g_0 : \mathbb{N} \to \mathbb{N}$ given by $g_0(n) = p \left\lceil \frac{n}{q} \right\rceil$ be the function defined in Section 1. For any $k \in \mathbb{N}$, we define $g_k : \mathbb{N} \to \mathbb{N}$ by $g_k(x) = g_0(x) + kp$. Furthermore, we define $u : \mathbb{N} \to \mathbb{N}$ by $u(x) = p \left\lfloor \frac{qx}{p^2} \right\rfloor$. We call each function g_k a *down-edge map* and u the *up-edge map*.

Using the maps above, we define $\widetilde{T}_{p/q}$ to be an infinite, labeled graph as follows. Let the vertex set be the multiples of p. For each vertex v of type i, we get edges of the form $v \to g_k(v)$ (when $g_k(v) \neq 0$) for all $k < b_i$, and we label each such edge by $r_i + kq$. This construction is basically the one given in [2] (and used in [9]), but that construction does not make use of the branch sequence.

As can be seen in [2], [9] and below, $\widetilde{T}_{p/q}$ is a tree and each function g_k maps a vertex to one of its children. The following proposition explains that u maps each vertex to its parent.

Proposition 2.1. *Suppose $v = pn$ where $n \in \mathbb{N}$ and v is of type i. Then $u(g_k(v)) = v$ for all $k < b_i$.*

Proof. Let $v = pn$ be of type i and $k < b_i$. Then $u(g_k(v))$ is given by

$$p\left\lfloor \frac{q}{p}\left(\frac{(p\lceil \frac{v}{q}\rceil + kp)}{p}\right)\right\rfloor = p\left\lfloor \frac{q}{p}\left(\left\lceil \frac{v}{q}\right\rceil + k\right)\right\rfloor = p\left\lfloor \frac{q}{p}\left(\frac{v+r_i}{q} + k\right)\right\rfloor$$

according to Corollary 1.8. Also

$$p\left\lfloor \frac{q}{p}\left(\frac{v+r_i}{q} + k\right)\right\rfloor = p\left\lfloor \frac{v}{p} + \frac{r_i + qk}{p}\right\rfloor = p\left\lfloor \frac{pn}{p} + \frac{r_i + qk}{p}\right\rfloor.$$

Consequently

$$u(g_k(v)) = p\left\lfloor \frac{pn}{p} + \frac{r_i + qk}{p}\right\rfloor = p\left(n + \left\lfloor \frac{r_i + qk}{p}\right\rfloor\right) = pn = v.$$

where the third equality follows since $r_i + qk < p$ by Corollary 1.8. □

Notice that u is only a partial inverse for g_k as the previous result is type-dependent (and only applies for multiples of p).

Lemma 2.2. *Let v be a multiple of p. Assume that v is of type i and $k < b_i$. If $v = (0, v_1, ..., v_n)_{\frac{p}{q}}$, then $g_k(v) = (0, r_i + kq, v_1, ..., v_n)_{\frac{p}{q}}$.*

Proof. Since v is of type i, Corollary 1.8 implies that

$$g_0(v) = \frac{p}{q}(v + r_i) = (0, r_i, v_1, ..., v_n)_{\frac{p}{q}}.$$

The second equality follows from the fact that multiplication of a multiple of q by $\frac{p}{q}$ shifts the digits. Then

$$g_k(v) = g_0(v) + kp = (0, r_i, v_1, ..., v_n)_{\frac{p}{q}} + kp = (0, r_i + kq, v_1, ..., v_n)_{\frac{p}{q}},$$

where the last equality follows from Theorem 1.1. By Corollary 1.8, $r_i + kq < p$, and so we have the valid $\frac{p}{q}$-representation of $g_k(v)$. □

We will use Lemma 2.2 to show that $\widetilde{T}_{p/q}$ describes the $\frac{p}{q}$-representations of the multiples of p. Before we explain this, we prove the following theorem, which allows us to prove that $\widetilde{T}_{p/q}$ is a tree and that it can be constructed in a purely combinatorial manner.

Theorem 2.3. *Let v be a multiple of p that is of type i. Then*

$$g_{b_i - 1}(v) + p = g_0(v + p).$$

Proof. Since v is a multiple of p, we know $v = (0, v_1, ..., v_n)_{\frac{p}{q}}$. Let j be the minimal index with $v_j < p - q$. Since v is type i, $v + p$ is of type $i + 1$. Notice that

$$v + p = (0, v_1 + (p - q), v_2 + (p - q), ..., v_j + q, v_{j+1}, ..., v_n)_{\frac{p}{q}}$$

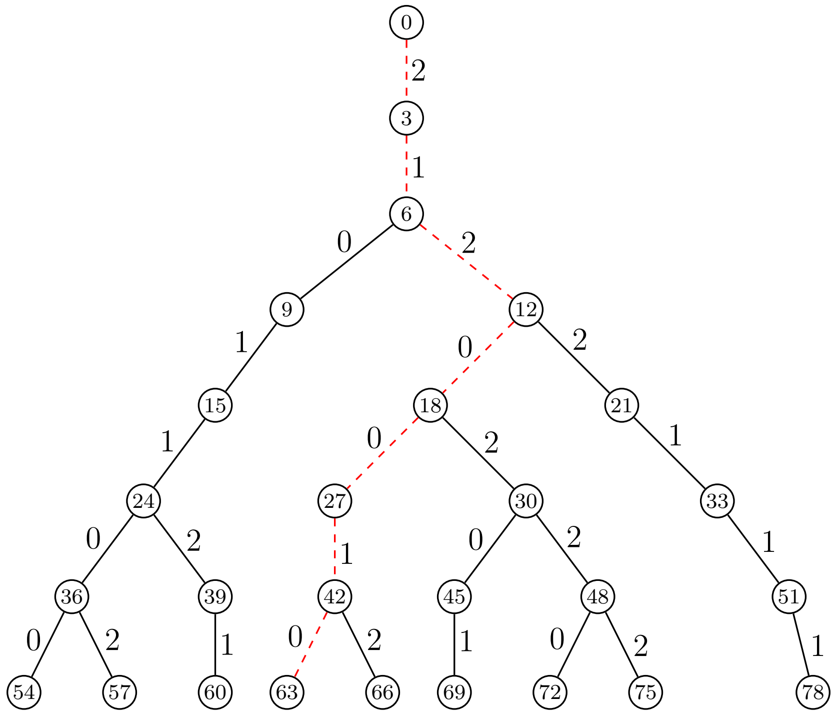

FIGURE 3. The $\frac{3}{2}$-representation tree.

by Theorem 1.1. By Lemma 2.2

$$g_0(v+p) = (0, r_{i+1}, v_1 + (p-q), v_2 + (p-q), ..., v_j + q, v_{j+1}, ..., v_n)_{\frac{p}{q}}$$

and

$$g_{b_i-1}(v) + p = (0, r_i + (b_i - 1)q, v_1, v_2, ..., v_n)_{\frac{p}{q}} + p.$$

Due to Corollary 1.8, we know $r_i + b_i q \geq p$; thus by Theorem 1.1

$$g_{b_i-1}(v)+p = (0, r_i+(b_i-1)q-(p-q), v_1+(p-q), v_2+(p-q), ..., v_j+q, v_{j+1}, ..., v_n)_{\frac{p}{q}}.$$

By Lemma 1.7, $r_{i+1} = r_i + qb_i - p$. Hence $g_0(v+p) = g_{b_i-1}(v) + p$. □

Theorem 2.3 implies that every multiple of p has exactly one incoming edge; therefore, $\widetilde{T}_{p/q}$ is an infinite, labeled, rooted tree with root 0. We call $\widetilde{T}_{p/q}$ the $\frac{p}{q}$*-representation tree*. See Figure 3 for a picture of the $\frac{3}{2}$-representation tree up to 78.

Since $\widetilde{T}_{p/q}$ is a rooted tree, there is a unique path from each vertex (i.e. multiple of p) to 0. Given a multiple of p, say v, we let $e_v = (e_1, \ldots, e_n)$ be the sequence of edge labels of the path from v to 0 in $\widetilde{T}_{p/q}$. We call e_v the *edge sequence for* v. The edge sequences encode the $\frac{p}{q}$-representations for multiples of p.

Theorem 2.4. *Let v be a multiple of p with $e_v = (e_1, \dots, e_n)$ the corresponding edge sequence. Then the $\frac{p}{q}$-representation of v is*

$$v = (0, e_1, \dots, e_n)_{\frac{p}{q}}.$$

Proof. This follows by induction on path length using Lemma 2.2. □

Corollary 2.5. *Let $m \in \mathbb{N}$. Then the $\frac{p}{q}$-representation of m is*

$$m = (u, e_1, \dots, e_n)_{\frac{p}{q}},$$

where u is the unique integer $0 \le u < p$ satisfying $m = ip + u$ and $(e_1, \dots, e_n)$ is the edge sequence for ip.

For example, we use Figure 3 to see that since $65 = 63 + 2$, we know $65 = (2, 0, 1, 0, 0, 2, 1, 2)_{\frac{3}{2}}$, which is shown by the red, dashed lines.

Corollary 2.6. *Let $a \in \mathbb{N}$ with $a \ge 3$ and $m \in \mathbb{N}$ with $m \ge a(a-1)$. The $\frac{a}{a-1}$-representation of m is*

$$m = (m_0, m_1, \dots, m_i, 1, 2, 3, \dots, a-2, a-1)_{\frac{a}{a-1}}.$$

Proof. According to the construction of the branch tree (see Figure 2), we have that the $\frac{a}{a-1}$-branch sequence is $(1, 1, \dots, 1, 2)$ and so the $\frac{a}{a-1}$-tree begins with $a-1$ single edges labeled by the cobranch sequence $(a-1, a-2, \dots, 2, 1)$. □

Corollary 2.7. *For all $n \in \mathbb{N}$, n is a multiple of q if and only if $n = (e_1, \dots, e_n)_{\frac{p}{q}}$ where $(e_1, \dots, e_n)$ is some edge sequence in $\widetilde{T}_{p/q}$.*

Proof. Let $n = qk$ for some $k \in \mathbb{N}$ with $n = (n_0, n_1, \dots, n_m)_{\frac{p}{q}}$. Now, $pk = \frac{p}{q}n = (0, n_0, n_1, \dots, n_m)_{\frac{p}{q}}$. Since pk is a vertex of $\widetilde{T}_{p/q}$, the edge sequence of the path from pk to 0 is $(n_0, n_1, \dots, n_m)$. The other direction is similar. □

For instance, $(2, 2, 0, 2, 1, 2)$ is an edge sequence in $\widetilde{T}_{3/2}$ (see Figure 3), and so $(2, 2, 0, 2, 1, 2)_{\frac{3}{2}}$ is the $\frac{3}{2}$-representation of a number that is divisible by 2 (it is equal to 32).

If $qn = (n_0, n_1, \dots, n_k)_{\frac{p}{q}}$, then $n = \sum_{i=0}^{k} \frac{n_i}{q} \left(\frac{p}{q}\right)^i$; the natural number representations of the latter type are precisely those considered in [2], [9], and [17]. Thus, those representations, called the $\frac{1}{q}\frac{p}{q}$-representations, correspond to the $\frac{p}{q}$-representations of the multiples of q, and Corollary 2.7 explains why the tree $\widetilde{T}_{p/q}$ is relevant for both types of representations.

In [2], [9], and above, the construction of the tree $\widetilde{T}_{p/q}$ relies heavily on the down-edge maps, which are only partially defined maps. It turns out that these trees can be constructed purely using the combinatorial data in the branch sequence. As noted a similar construction can be found in [16].

We define $T_{p/q}$ be the infinite, labeled, rooted tree with vertex set the multiples of p, root 0, and edges constructed in stages as follows.

Stage 1: Connect 0 to each vertex in $\{p, 2p, \ldots, (b_q - 1)p\}$.

General stage: Let v be the minimal vertex with no outgoing edges. If v is of type s, then connect v to the smallest b_s vertices that, at this stage, have no incoming edges.

Note that $T_{p/q}$ is a rooted tree as each vertex (except for 0) has exactly one incoming edge. Each vertex of type s has exactly b_s children (and hence b_s outgoing edges). For $v = ip$, we label the incoming edge to v by $iq \pmod{p}$.

For instance, Figure 3 also shows the tree $T_{3/2}$. The $\frac{3}{2}$-branch sequence is $(1, 2)$ so we first connect 0 to the $b_2 - 1 = 2 - 1 = 1$ vertex in $\{3\}$. We then traverse the newly added vertices alternating between adding a single branch and a double branch. Each edge is labeled by the successive multiples of 2 mod 3. It is not a coincidence that the trees $T_{3/2}$ and $\widetilde{T}_{3/2}$ are the same.

Theorem 2.8. *The two infinite, labeled, rooted trees $T_{p/q}$ and $\widetilde{T}_{p/q}$ are identical.*

Proof. By construction, both trees have vertex set equal to the multiples of p and both trees have 0 as the root. Moreover, we see that 0 has edges to the set of vertices $\{p, 2p, \ldots, (b_q - 1)p\}$ in both cases. By definition, if v is of type s, then $g_{k+1}(v) = g_k(v) + p$ when $k < b_s - 1$ and by Theorem 2.3 $g_{b_s - 1}(v) + p = g_0(v + p)$. The result then follows by induction on the multiples of p; in particular, if we suppose the two trees coincide up to the vertex ip, then the previous two facts imply that the trees also agree on the vertex $ip + p = (i + 1)p$. Finally, note that the first edge label (on the edge connecting 0 to p) in both trees is $q = r_q + q$, since $r_q = 0$. Then, by Theorem 2.3 and Corollary 1.8, each successive edge of $\widetilde{T}_{p/q}$ is labeled by the next multiple of $q \pmod{p}$, which matches $T_{p/q}$. $\square$

In practice, constructing $T_{p/q}$ is a simple process and both vertex and edge labeling can be done after construction. After the first stage (which is special) we periodically cycle through the branch sequence: when we are at entry b in the branch sequence, we populate the oldest constructed vertex that has no outgoing edges with b outgoing edges to b new vertices. Finally, we label the edges by multiples of q mod p and the vertices by the multiples of p in the order they were added to the tree. For instance, Figure 4 show the first seven stages of constructing $T_{5/3}$ whose branch sequence is $(2,1,2)$. In the last image, we add the vertex labels (i.e. the multiples of 5) and the edge labels, which are obtained by cycling through $3,1,4,2,0$ (i.e. the multiples of 3 mod 5). A slightly more complicated example is pictured in Figure 5 where we use the $\frac{9}{4}$-branch sequence $(2,2,2,3)$.

Proposition 2.9. *Let $n = (n_0, n_1, ..., n_k)_{\frac{p}{q}}$. Then $n_0 = n \pmod p$ and for $i > 0$, $n_i = q\left(\frac{u^{i-1}(v)}{p}\right) \pmod p$ where $v = p\left\lfloor \frac{n}{p} \right\rfloor$ and u is the up-edge map.*

Proof. By Proposition 2.1 and the construction of $\widetilde{T}_{p/q}$, $u(v)$ is the unique vertex with an edge to v. Corollary 2.5 then implies that n_i is the edge label of the incoming edge of $w = u^{i-1}(v)$. Finally, the construction of $T_{p/q}$ shows this edge label is $q\left(\frac{w}{p}\right) \pmod p$. $\square$

We finish this section by providing a formula that counts the number of nodes at any level in $T_{p/q}$; for examples of these sequences see A073941, A072493 and A120160 in [1]; this result answers a question found in A073941.

Theorem 2.10. *The number of vertices of distance n from the root in $T_{p/q}$ is given by a_n where $a_0 = 1$ and for $n \geq 1$*

$$a_n = \left\lceil \frac{(p-q)}{q} \cdot \sum_{i=0}^{n-1} a_i \right\rceil.$$

Proof. Let $(k_1, k_2, k_3, \ldots)$ be the sequence of knockout carries as defined in Proposition 1.5. Then, by that proposition and the construction of $\widetilde{T}_{p/q}$, k_j is the the minimal vertex of distance j from 0. Therefore, for $n \geq 1$ we know that $k_n = p\sum_{i=0}^{n-1} a_i$, and this implies

$$a_n = \frac{k_{n+1} - k_n}{p}.$$

Recall $k_{n+1} = g_0(k_n) = p\left\lceil \frac{k_n}{q} \right\rceil$ and assume that k_n is of type i. Then

$$a_n = \left\lceil \frac{k_n}{q} \right\rceil - \frac{k_n}{p} = \frac{k_n + r_i}{q} - \frac{k_n}{p} = \frac{k_n(p-q)}{pq} + \frac{r_i}{q},$$

and so

$$a_n = \frac{(p-q)\sum_{i=0}^{n-1} a_i}{q} + \frac{r_i}{q}.$$

Finally, since $0 \leq r_i < q$, we conclude $a_n = \left\lceil \frac{(p-q)}{q} \cdot \sum_{i=0}^{n-1} a_i \right\rceil$. $\square$

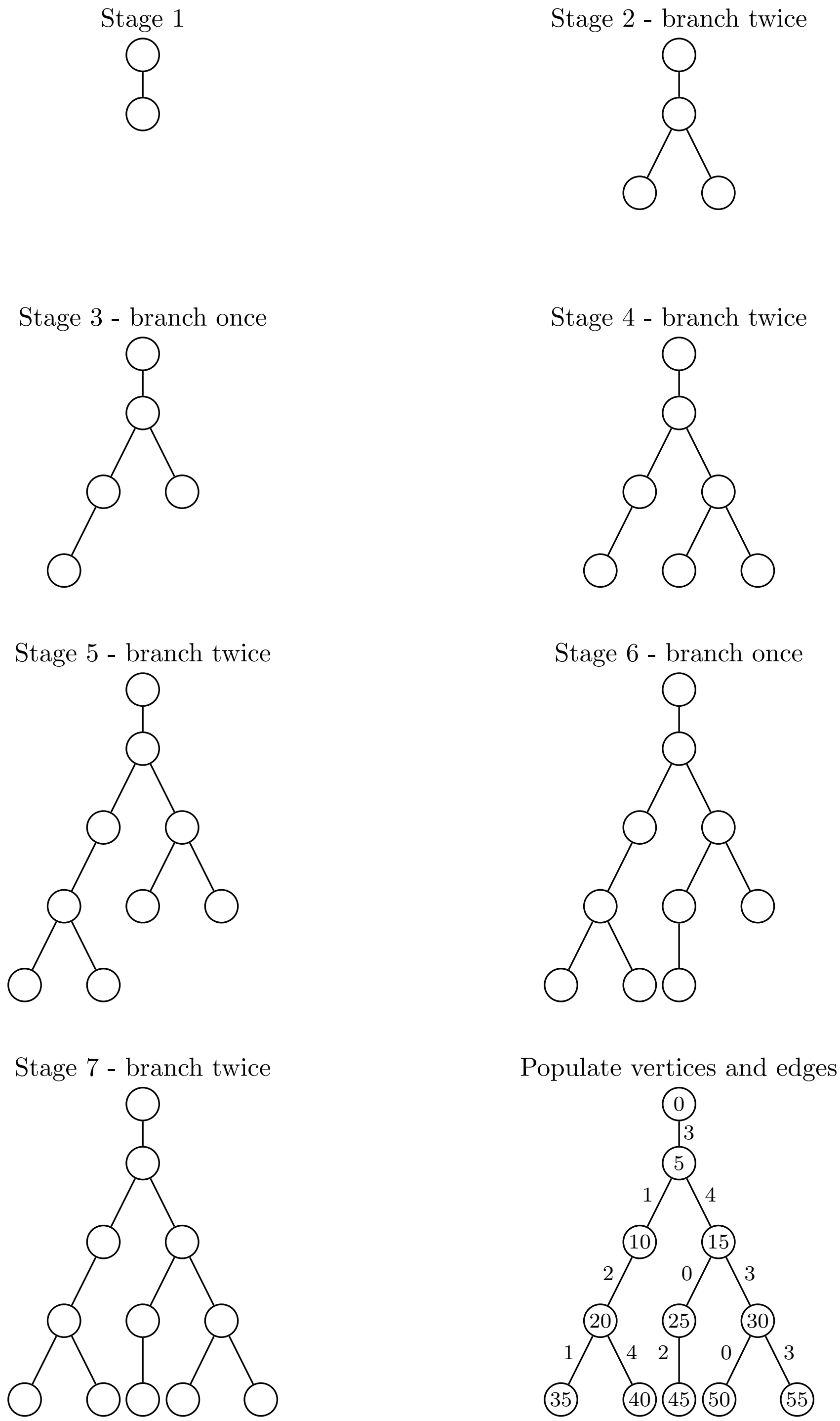

FIGURE 4. Seven stages of generating $T_{5/3}$ with branch sequence $(2, 1, 2)$ labeled accordingly.

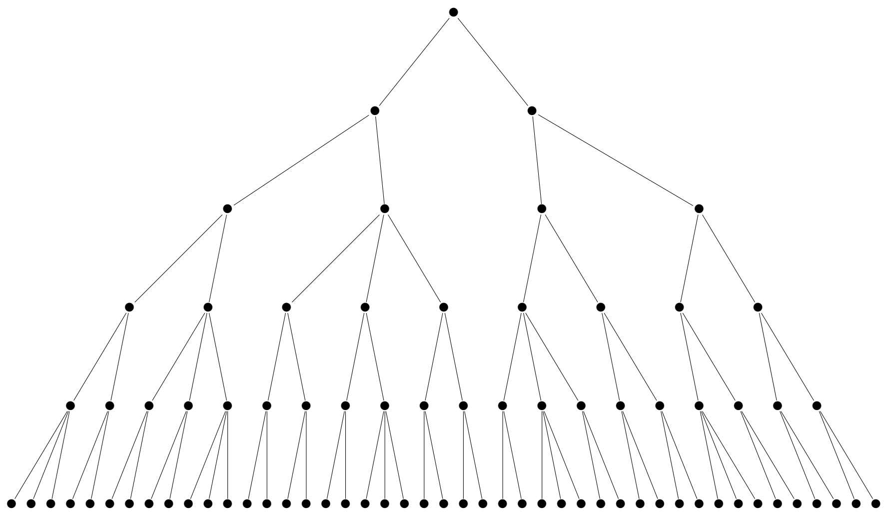

FIGURE 5. The tree $T_{9/4}$ generated by the $\frac{9}{4}$-branch sequence $(2, 2, 2, 3)$. The vertices should be labeled with the multiples of 9, top to bottom and left to right. Similarly, the edges should be labeled by cycling through $4, 8, 3, 7, 2, 6, 1, 5, 0$.

We add to Proposition 1.5 by combining Theorem 2.10 and Proposition 1.5 to obtain an alternate formula for the number of positive integers whose $\frac{p}{q}$-representations have length n.

Corollary 2.11. *For $n \geq 1$, the number of positive integers with $\frac{p}{q}$-representations of length n, denoted c_n, is given by $c_n = pa_n$ where $(a_1, a_2, a_3, \ldots)$ is the sequence described in the previous theorem.*

To demonstrate the last two results, we refer to Figure 5, which shows the tree $T_{9/4}$. Using Theorem 2.10, we can compute that levels 0-5 contain $1, 2, 4, 9, 20$ and 45 vertices respectively. Therefore, there are $180 = 9 \cdot 20$ natural numbers whose $\frac{9}{4}$-representations are length 4.

3. COMBINATORICS OF $\frac{p}{q}$-REPRESENTATIONS

The results from the previous sections allow us to partially generalize the results of [4] and [5] to $\frac{p}{q}$-representations. We proceed to investigate some of the combinatorics associated with $\frac{p}{q}$-representations. As in the previous sections, we fix $p, q \in \mathbb{N}$, with $p > q \geq 1$ and $\gcd(p, q) = 1$.

If $n = (n_0, n_1, \ldots, n_k)_{\frac{p}{q}}$ and $m = (m_0, m_1, \ldots, m_l)_{\frac{p}{q}}$ are both natural numbers, we say $n \ll_{\frac{p}{q}} m$ when $n_i \leq m_i$ for all i (recall that $n_i = 0$ for $i > k$ and $m_i = 0$ for $i > l$), and we say m dominates n in base-$\frac{p}{q}$.

Since $\frac{p}{q}$-representations are unique, $\ll_{\frac{p}{q}}$ is a partial order on $\mathbb{N}$, which we call the $\frac{p}{q}$-dominance order (see [19] for more information about partial orders). When $q = 1$, dominance order is fully understood by the results in [5]; the poset is a graded lattice with rank function given by the sum-of-digits function. Unfortunately, when $q \neq 1$, the poset satisfies none of these properties (see Figure 6). Nonetheless, there is a connection to the system of base-$\frac{p}{q}$ arithmetic described in Section 1.

Theorem 3.1. *For $n, m \in \mathbb{N}$, $n \ll_{\frac{p}{q}} n+m$ if and only if* $\displaystyle\sum_{i\geq 0} \epsilon_i^{\frac{p}{q},n,m} = 0.$

Proof. Let $n = (n_0, n_1, \ldots, n_k)_{\frac{p}{q}}$ and $m = (m_0, m_1, \ldots, m_l)_{\frac{p}{q}}$ both be natural numbers. First, we assume that the sum of carries is nonzero and that i is the minimal index with $\epsilon_i \neq 0$. By definition,

$$\epsilon_i = \left\lfloor \frac{n_i + m_i + q\epsilon_{i-1}}{p} \right\rfloor.$$

Since $\epsilon_{i-1} = 0$, we have

$$\epsilon_i = \left\lfloor \frac{n_i + m_i}{p} \right\rfloor < \left\lfloor \frac{2p}{p} \right\rfloor = 2.$$

Therefore $\epsilon_i = 1$. Since $m_i < p$, following Equation (1), we know

$$(n+m)_i = n_i + m_i - p < n_i + p - p = n_i.$$

Hence, $(n+m)_i < n_i$ and so $n \not\ll_{\frac{p}{q}} n+m$. We conclude that $n \ll_{\frac{p}{q}} n+m$ implies $\sum_{i\geq 0} \epsilon_i = 0$.

For the other direction, we assume $\sum_{i\geq 0} \epsilon_i = 0$, which this means $\epsilon_i = 0$ for all i. Then, for each i, we see

$$(n+m)_i = n_i + m_i + q(0) - p(0) = n_i + m_i.$$

However, $m_i \geq 0$ so that $(n+m)_i \geq n_i$. By definition, $n \ll_{\frac{p}{q}} n+m$. $\square$

In order to generalize the results in [4], we first need to introduce the following important sequence generated using the tree $T_{p/q}$ and the associated down-edge map g_0. Let $(h_1, h_2, h_3 \ldots)$ be the sequence given by $h_1 = 1$ and

$$h_n = \begin{cases} h_i + 1 & \text{if } np = g_0(ip) \text{ for some } i, \\ 1 & \text{otherwise.} \end{cases}$$

We can construct this sequence using an alternate labeling of the tree $T_{p/q}$ as follows. We label the root, 0, by 0. Next, if a vertex has been labeled m, we label the minimal child (i.e. the first created during construction of $T_{p/q}$) by $m+1$ and label the other children by 1. The

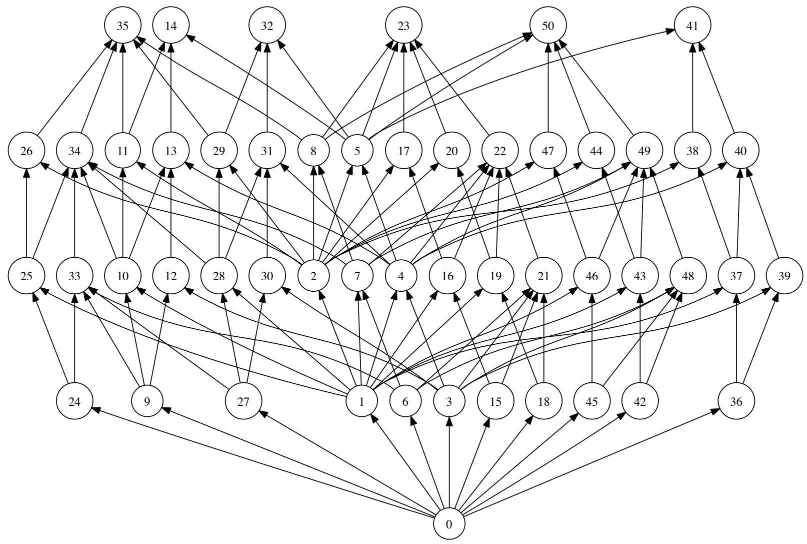

FIGURE 6. The Hasse diagram of $\frac{3}{2}$-dominance order truncated at 50 created using Sage ([20]).

entry h_i corresponds to the label attached to the vertex ip. See Figure 7 for this labeling of $T_{3/2}$ (compare to Figure 3). A closed form for this sequence comes from the sum-of-digits function.

Theorem 3.2. *The sequence* $(h_1, h_2, h_3, \ldots)$ *defined recursively above is given by*

$$h_i = \frac{1}{p-q}\left(p + \mathrm{sum}_{\frac{p}{q}}((i-1)p) - \mathrm{sum}_{\frac{p}{q}}(ip)\right).$$

Proof. Let $(y_1, y_2, y_3, \ldots)$ be the sequence given by

$$y_i = \frac{1}{p-q}\left(p + \mathrm{sum}_{\frac{p}{q}}((i-1)p) - \mathrm{sum}_{\frac{p}{q}}(ip)\right).$$

Without loss of generality, we assume that ip is of type $s+1$, which means $(i-1)p$ is of type s. Now, let $ip = (0, e_1, \ldots, e_k)_{\frac{p}{q}}$ and $(i-1)p = (0, e'_1, \ldots, e'_t)_{\frac{p}{q}}$. According to Theorem 2.3 and Lemma 2.2,

$$g_0(ip) - p = g_{b_s-1}(ip-p) = (0, r_s + (b_s-1)q, e'_1, \ldots, e'_t)_{\frac{p}{q}}$$

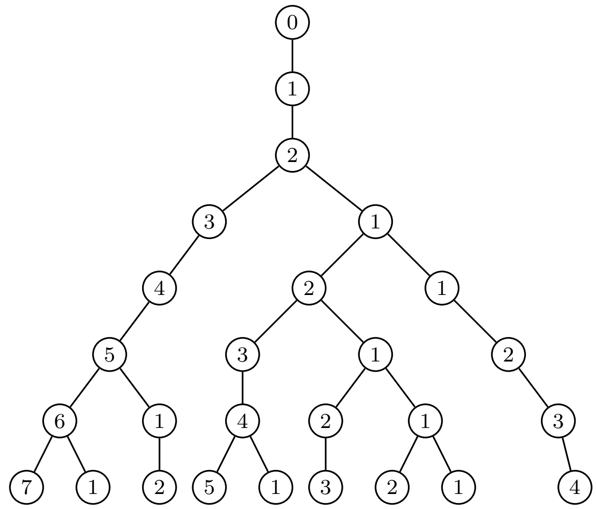

FIGURE 7. The $\frac{3}{2}$-representation tree with alternate labeling. The corresponding sequence $(h_1, h_2, \ldots)$ is $(1,2,3,1,4,2,1,5,3,1,2,6,1,4,2,1,3,7,1,2,5,1,3,2,1,4,\ldots)$; see A087088 in [1].

and $g_0(ip) = (0, r_{s+1}, e_1, \ldots, e_k)_{\frac{p}{q}}$. Hence

$$\begin{aligned}\operatorname{sum}_{\frac{p}{q}}(g_0(ip)-p) - \operatorname{sum}_{\frac{p}{q}}(g_0(ip)) &= \operatorname{sum}_{\frac{p}{q}}(g_{b_s-1}(ip-p)) - \operatorname{sum}_{\frac{p}{q}}(g_0(ip)) \\ &= \sum_{i\geq 0} e'_i + (b_s-1)q + r_s - \sum_{i\geq 0} e_i - r_{s+1} \\ &= \sum_{i\geq 0} e'_i - \sum_{i\geq 0} e_i + b_s q - q + r_s - r_{s+1} \\ &= \operatorname{sum}_{\frac{p}{q}}((i-1)p) - \operatorname{sum}_{\frac{p}{q}}(ip) + p - q,\end{aligned}$$

where the last equality follows from Lemma 1.7. Thus, if $np = g_0(ip)$, then

$$\begin{aligned}y_n &= \frac{1}{p-q}\Big(p + \operatorname{sum}_{\frac{p}{q}}(np-p) - \operatorname{sum}_{\frac{p}{q}}(n)\Big) \\ &= \frac{1}{p-q}\Big(p + \operatorname{sum}_{\frac{p}{q}}((i-1)p) - \operatorname{sum}_{\frac{p}{q}}(ip) + p - q\Big) \\ &= y_i + \frac{p-q}{p-q} = y_i + 1.\end{aligned}$$

Next, if $np \neq g_0(ip)$ for any i, then $u(np) = u((n-1)p)$ where u is the up-edge map, i.e. np and $(n-1)p$ have the same parent in $T_{p/q}$. Theorem 2.4 implies that we must have $np = (0, e_1, e_2, \ldots, e_k)_{\frac{p}{q}}$ and $(n-1)p = (0, d_1, e_2, \ldots, e_k)_{\frac{p}{q}}$; moreover, $e_1 = d_1 + q$ by construction of

$T_{p/q}$. Therefore,

$$\begin{aligned} y_n &= \frac{1}{p-q}\left(p + \mathrm{sum}_{\frac{p}{q}}((n-1)p) - \mathrm{sum}_{\frac{p}{q}}(n)\right) \\ &= \frac{1}{p-q}\left(p + d_1 - e_1\right) = 1. \end{aligned}$$

Since $(y_1, y_2, y_3, \ldots)$ satisfies the same recurrence as $(h_1, h_2, h_3, \ldots)$ and $y_1 = 1 = h_1$, the two sequences must be the same. $\square$

In the case where $q = 1$, this sequence is sometimes referred to as a ruler sequence, and in this case, h_n gives the largest power of p dividing pn. See A001511, A051064, or A115362 in [1] for examples. We use this sequence to construct another sequence, which corresponds to the p-adic valuation when $q = 1$. Let $(v_1, v_2, v_3, \ldots)$ be given by

$$v_i = \begin{cases} h_{\frac{i}{p}} & \text{if } i \text{ is a multiple of } p \\ 0 & \text{otherwise.} \end{cases}$$

For instance, the corresponding sequence for the $\frac{3}{2}$-representations according to Figure 7 is

$$(0, 0, 1, 0, 0, 2, 0, 0, 3, 0, 0, 1, 0, 0, 4, 0, 0, 2, 0, 0, 1, 0, 0, 5, 0, 0, 3, \ldots),$$

and the sequence for $\frac{2}{1}$-representations is

$$(0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, 2, 0, 1, 0, 4, 0, 1, 0, 2, 0, 1, 0, 3, 0, 1, 0, \ldots),$$

which is the 2-adic valuation (see in A007814 [1]). The sequence $(v_1, v_2, v_3, \ldots)$ leads us to a result similar to Kummer's Theorem (see [4]) for $\frac{p}{q}$-representations.

For any sequence of nonzero natural numbers, C, we define the C-factorial by

$$(m!)_C = \begin{cases} C_m C_{m-1} \cdots C_1 & \text{when } m \neq 0 \\ 1 & \text{when } m = 0. \end{cases}$$

We use the C-factorial to construct generalized binomial coefficients associated to C, called the C-nomial coefficients, which are given by

$$\binom{m}{n}_C = \begin{cases} \frac{(m!)_C}{(n!)_C((m-n)!)_C} & \text{when } 0 \leq n \leq m \\ 0 & \text{otherwise.} \end{cases}$$

The previous construction works for any sequence of nonzero integers, but the generalized binomial coefficients cannot be expected to be integers (see [13]).

With these ideas in hand, we construct one more sequence of interest $W^{\frac{p}{q}} = (W_1, W_2, W_3 \ldots)$ by

$$W_n = p^{v_n},$$

where $(v_1, v_2, v_3, \ldots)$ is defined above and described in Theorem 3.2. For example, when $\frac{p}{q} = \frac{3}{2}$, we have

$$W^{\frac{3}{2}} = (1, 1, 3, 1, 1, 9, 1, 1, 27, 1, 1, 3, 1, 1, 81, 1, 1, 9, 1, 1, 3 \ldots).$$

In place of $(m!)_W$ we use $(m!)_{\frac{p}{q}}$, and in place of $\binom{m}{n}_W$ we use $\binom{m}{n}_{\frac{p}{q}}$. We call the latter numbers the $\frac{p}{q}$-binomial coefficients.

Lemma 3.3. *For any $n \in \mathbb{N}$*

$$(n!)_{\frac{p}{q}} = p^{\frac{1}{p-q}(n - \mathrm{sum}_{\frac{p}{q}}(n))}.$$

Proof. Let $n \in \mathbb{N}$ such that $n = mp + u$ with $0 \le u < p$. By definition we see that

$$(n!)_{\frac{p}{q}} = \prod_{i=1}^{n} W_i = p^{\sum_{1 \le ip \le n} h_i}.$$

Now,

$$\sum_{1 \le ip \le n} h_i = \frac{1}{p-q} \sum_{1 \le ip \le n} \Big(p + \mathrm{sum}_{\frac{p}{q}}((i-1)p) - \mathrm{sum}_{\frac{p}{q}}(ip) \Big).$$

This sum telescopes, and so

$$\begin{aligned}(p-q) \cdot \sum_{1 \le ip \le n} h_i &= mp + \mathrm{sum}_{\frac{p}{q}}(0) - \mathrm{sum}_{\frac{p}{q}}(mp) \\ &= mp + u - u - \mathrm{sum}_{\frac{p}{q}}(mp) = n - \mathrm{sum}_{\frac{p}{q}}(n).\end{aligned}$$

The result follows. □

Using the $\frac{p}{q}$-binomial coefficients, we obtain the following generalization of the main theorem in [4], which, in turn, is an analog of Kummer's Theorem.

Theorem 3.4. *Let $m, n \in \mathbb{N}$ with $n \le m$. Then the highest power of p that divides $\binom{m}{n}_{\frac{p}{q}}$ is the total number of carries when adding the $\frac{p}{q}$-representations of n and $m - n$.*

Proof. As $\binom{m}{n}_{\frac{p}{q}} = \frac{(m!)_{\frac{p}{q}}}{(n!)_{\frac{p}{q}}((m-n)!)_{\frac{p}{q}}}$, we apply Lemma 3.3 and simplify:

$$\binom{m}{n}_{\frac{p}{q}} = p^{\left(\frac{1}{p-q}\right)(\mathrm{sum}_{\frac{p}{q}}(m-n) + \mathrm{sum}_{\frac{p}{q}}(n) - \mathrm{sum}_{\frac{p}{q}}(m))}.$$

Now the result follows by Proposition 1.4. □

We pair Theorem 3.4 with Theorem 3.1 to get a characterization of $\frac{p}{q}$-dominance in terms of the $\frac{p}{q}$-binomial coefficients.

Corollary 3.5. *For all $n, m \in \mathbb{N}$ with $n \leq m$, we have $n \ll_{\frac{p}{q}} m$ if and only if $\binom{m}{n}_{\frac{p}{q}} \not\equiv 0 \pmod{p}$.*

4. Future Investigations

In this final section, we mention a few future directions for study of the combinatorics associated with $\frac{p}{q}$-representations.

First, since rational base dominance orders are not as well-behaved as their integral base counterparts, it would be interesting to provide other characterizations of these posets, determine how the sum-of-digits function (see [17]) relates to the poset, or study other combinatorial properties as they relate to $\frac{p}{q}$-arithmetic. Every property of the base-b dominance orders discussed in [5] would be nice to understand for these orders. The poset does appear to have a rank function (in the most general sense of a rank function, see [19]) though it is not the sum-of-digits function. Finding a formula for the rank function would be beneficial; moreover, simply explaining the sequence of natural numbers only dominating zero, which we call the *rank-one elements*, is desired. For instance, in the $\frac{3}{2}$-dominance order, the rank-one elements are

$$1, 3, 6, 9, 15, 18, 24, 27, 36, 42, 45, 54, 63, 69, 81, \ldots.$$

Moreover, we would like to know if there is a combinatorial interpretation of $\binom{m}{n}_{\frac{p}{q}}$, defined in the previous section, in terms of the $\frac{p}{q}$-dominance order.

In addition to further understanding of the dominance order, there may be other interesting combinatorics associated with the tree $T_{p/q}$. In particular, can we use the results in Section 3 and associated sequences to find a closed form for the sequence of knockout carries $(k_1, k_2, k_3, \ldots)$ from Section 1?

Finally, in the case of $\frac{3}{2}$-representations, the down-edge map g_0 is related to the famous Collatz conjecture. We imagine some of the difficulty in understanding the $\frac{3}{2}$-dominance order is related to the difficulty of solving the Collatz conjecture. Finding an interpretation of the Collatz conjecture in terms of $\frac{3}{2}$-representations, the $\frac{3}{2}$-representation tree, or the $\frac{3}{2}$-dominance order would be fascinating and lead to further interesting questions in this situation.

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