An Introduction to Posets and Möbius Inversion

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Introduction

Theory of Finite Differences

Let $f : \mathbb{Z}_{\geq 0} :\rightarrow \mathbb{R}$. In other words, f defines a sequence

$$\{f(m)\}_{m=0}^{\infty} = \{f(0), f(1), f(2), ...\}.$$

There is an analogue of the integral operator on this space

$$\mathcal{I}(f(n)) = \sum_{i=0}^{n} f(i)$$

Also, there is an analogue of the derivative operator:

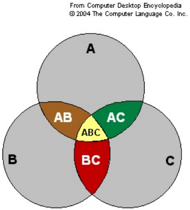
$$\Delta(f(n)) = f(n) - f(n-1); \quad (f(-1) := 0)$$

Fundamental Theorem of Difference Calculus:

$$\Delta\mathcal{I}(f(n)) = f(n).$$

Principle of Inclusion/Exclusion

Suppose U is a set with A, B, C all subsets of U.



$$|U-(A\cup B\cup C)| = |U|-|A|-|B|-|C|+|A\cap B|+|A\cap C|+|B\cap C|-|A\cap B\cap C|$$

General Case: Let M be a set with subsets $M_1, M_2, ..., M_n$. Then

$$\left|M-\bigcup_{i=1}^{n} M_i\right| = |M|-\sum_{i=1}^{n}|M_i|+\sum_{1\leq i\leq j\leq n}|M_i\cap M_j|+\cdots+(-1)^n\left|\bigcap_{i=1}^{n} M_i\right|$$

Number Theory

Let $\mathbb{Z}_{>0}$ be the set of all positive integers. We define a function $\mu : \mathbb{Z}_{>0} \to \{-1, 0, 1\}$ in the following way:

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct prime factors} \\ 0 & \text{if } n \text{ has one or more repeated prime factors} \end{cases}$$

Theorem: Suppose that f and g are two functions from $\mathbb{Z}_{>0} \to \mathbb{R}$ satisfying $f(n) = \sum_{d|n} g(d)$ for all n. Then

$$g(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d)$$

Unrelated Examples?

- Finite Series (Difference Calculus):

$$\Delta\mathcal{I}(f(n)) = f(n)$$

- Combinatorics (Probability):

$$\left| M - \bigcup_{i=1}^{n} M_i \right| = \sum_{T \subset \{1,\ldots,n\}} (-1)^{|T|} \left| \bigcap_{i \in T} M_i \right|$$

- Number Theory:

$$g(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) f(d)$$

Posets

Definition

Given a set P, we say a binary relation $\leq$ is a **partial order** on P if it satisfies three properties:

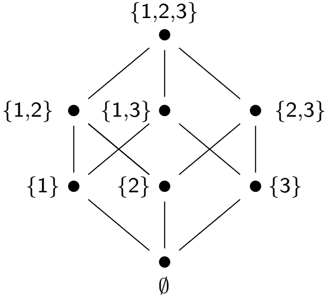
- Reflexivity: $a \leq a$ for all $a \in P$.
- Antisymmetry: If $a \leq b$ and $b \leq a$ then $a = b$.
- Transitivity: If $a \leq b$ and $b \leq c$ then $a \leq c$.

A set with a partial order is called a **poset**.

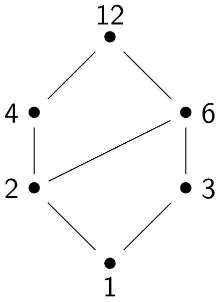
Examples of Posets

Example ($\mathbb{Z}_{\geq 0}$)

Example (Subsets)



Example (Divisors)



Incidence Algebras on Posets

$(P, \leq)$ is a poset. We define an **interval**:

$$[x, z] = \{\mathbf{y} \in P | x \leq \mathbf{y} \leq z\}$$

Incidence algebra:

$$\begin{aligned} I(P) &= \{f : P \times P \to \mathbb{R} |\ f(x, y) = 0 \text{ unless } x \leq y\} \\ &= \{f : \text{intervals of } P \to \mathbb{R}\} \end{aligned}$$

- Characteristic function in $I(P)$: $e(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$

- Defining function of P in $I(P)$: $\zeta(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$

Incidence Algebras on Posets

$I(P)$ has two nice operations:

- Addition: pointwise

$$(f+g)(x,z) = f(x,z) + g(x,z)$$

- Multiplication: convolution

$$(f*g)(x,z) = \sum_{x \le y \le z} f(x,y)g(y,z)$$

Identity for Multiplication: $e(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$

Proposition

A function $f \in I(P)$ is invertible if and only if $f(x,x) \neq 0$ for all $x \in P$.

The Möbius Function of a Poset

Recall: $\zeta(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$

Question: What is ζ^{-1}?

Answer: The Möbius function, μ.

Recursive Formula:

$$\mu(x, x) = 1$$

$$\mu(x, z) = -\sum_{x \leq y < z} \mu(x, y)$$

Möbius Inversion Theorem

Theorem

Let $(P, \leq)$ be a (finite) poset and let f, g be elements of $I(P)$. The following are equivalent:

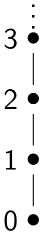
$$(a) \quad f(x,z) = \sum_{x \leq y \leq z} g(x,y) \qquad (f = g * \zeta)$$

$$(b) \quad g(x,z) = \sum_{x \leq y \leq z} f(x,y)\mu(y,z) \qquad (g = f * \mu)$$

where μ is the Möbius function for the poset.

Theory of Finite Differences

The Chain: $\mathbb{Z}_{\geq 0}$



$$\mu(i,j) = \begin{cases} 1 & j = i \\ -1 & j - i = 1 \\ 0 & \text{otherwise} \end{cases}$$

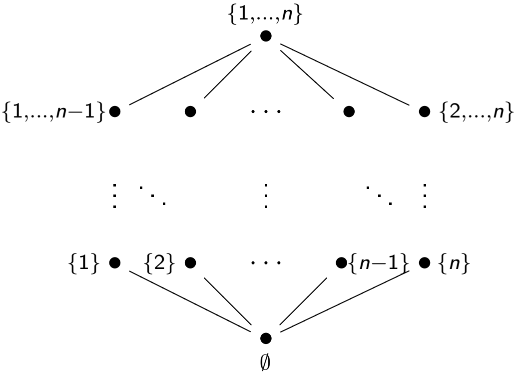
Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$. Recall $\mathcal{I}(f(n)) = \sum_{i=0}^{n} f(i)$.

Möbius Inversion:

$$f(n) = \sum_{i=0}^{n} \mathcal{I}(f(i))\mu(i,n) = \mathcal{I}(f(n)) - \mathcal{I}(f(n-1)) = \Delta\mathcal{I}(f(n))$$

Principle of Inclusion/Exclusion

Subsets of $S = \{1, 2, ..., n\}$

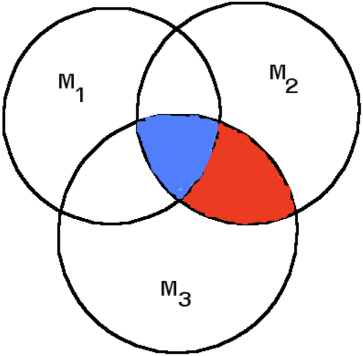


$$\mu(S,T) = \begin{cases} (-1)^{|T|-|S|} & S \subseteq T \\ 0 & \text{otherwise} \end{cases}$$

Principle of Inclusion/Exclusion

M a set with subsets $M_1, M_2, ..., M_n$.
Let $(P, \subseteq)$ be the poset of subsets of $S = \{1, ...n\}$.
We define two functions on the poset $(P, \subseteq)$:

- $f(V) = \left| \bigcap_{i \in V} M_i \right|$
- $g(V) = \left| \{a \in M | \; a \in M_i \; \forall i \in V; \; a \notin M_j \; \forall j \notin V\} \right|$



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$g(\{2,3\}) =$ Red

$g(\{1,2,3\}) =$ Blue

Note: $f(\{2,3\}) = g(\{2,3\}) + g(\{1,2,3\})$

Principle of Inclusion/Exclusion

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Let $(P, \subseteq)$ be the poset of subsets of $S = \{1, ...n\}$.
We define two functions on the poset $(P, \subseteq)$:

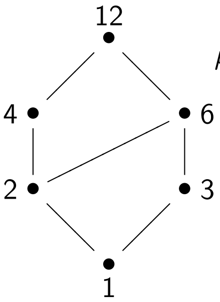
- $f(V) = \left| \bigcap_{i \in V} M_i \right|$
- $g(V) = \left| \{a \in M |\ a \in M_i\ \forall i \in V;\ a \notin M_j\ \forall j \notin V\} \right|$
- $f(V) = \sum_{V \subseteq T} g(T)$
- Remark: $g(\emptyset) = \left| M - \bigcup_{i=1}^{n} M_i \right|$

Möbius Inversion:

$$\left| M - \bigcup_{i=1}^{n} M_i \right| = g(\emptyset) = \sum_{\emptyset \subseteq T} \mu(\emptyset, T) f(T)$$

$$= \sum_{T} (-1)^{|T|} \left| \bigcap_{i \in T} M_i \right|$$

Number Theory



$$\mu(1,n) = \begin{cases} (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes} \\ 0 & \text{if } n \text{ has repeated prime factors} \end{cases}$$

Remark: $\mu(d,n) = \mu\left(1, \frac{n}{d}\right)$

$f, g : \mathbb{Z}_{>0} \to \mathbb{R}$ with $f(n) = \sum_{d \leq n} g(d)$

Möbius Inversion:

$$g(n) = \sum_{d \leq n} \mu(d,n) f(d) = \sum_{d \leq n} \mu\left(1, \frac{n}{d}\right) f(d) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) f(d)$$

Thanks!

Any Questions?

References

Bender, E.A and J.R. Goldman, *On the Applications of Mobius Inversion in Combinatorial Analysis*. The American Mathematical Monthly, Vol. 82, No. 8. (Oct., 1975), pp. 789-803.

Cameron, Peter J., *Combinatorics: Topics, Techniques, Algorithms*. Cambridge University Press, 1994.