ROGER NELSEN'S BOOKS, SO FAR

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In the 1930s, a group of French mathematicians formed a secret society in order to write a series of rigorous mathematics texts under the collective pseudonym Nicolas Bourbaki that eschewed geometric intuition. While writing about the life of Donald Coxeter and his contributions to mathematics and "saving geometry," Siobhan Roberts [19] describes the Bourbaki group,

> [Bourbaki] wanted to overhaul the entire structure of mathematics. In so doing, he sought to stamp out the use of diagrams. Bourbaki endeavored to write an algebraic encyclopedia of mathematics without a single picture. This aversion to shapes was defended as serving the interest of purity: all mathematical results were to be reached by reason alone—by rationality—rather than by the corruptible visual sense. According to Bourbaki, visual perception of the world was unreliable, our eyes leaving us victim to subjectivity and error.

Roberts provides evidence of this by quoting one-time Bourbaki member Pierre Cartier, "Bourbaki made a point of no pictures. Rather it was based on pure logical reasoning, as little visual insight as possible. Visual insight was considered a concession to human weakness."

Ian Stewart [20] describes how Bourbakism has fallen and a paradigm shift has occurred in mathematics. However, Stewart also notes that "Bourbakism didn't just disappear." Indeed, many of the mathematics texts one encounters still suffers the fate of Bourbaki's influence. Open any number of undergraduate or graduate texts and count the number of diagrams or pictures you see; you will likely be underwhelmed.

In a blog post, Terence Tao [22] describes dividing mathematical education into three stages: pre-rigorous, rigorous, and post-rigorous. The first and last stages revolve around intuition—the first on uninformed intuition, examples, hand waving, etc., and the second using intuition supported by the rigorous foundation. He notes that "the point of rigour is not to destroy all intuition; instead, it should be used to destroy bad intuition while clarifying and elevating good intuition." His description helps a modern mathematician make sense of both Bourbakism and the intuitive mathematics that led to the Bourbaki movement. When we think of mathematics, we should have a notion of both formal logical and algebraic thought as well as informed geometric intuition. When used together, one is able to build a picture of the mathematical landscape that can inform educational and learning processes, as well as advance mathematical research.

As Bill Thurston [23] points out, the controversy around the computer-aided proof of the four-color theorem "reflected a continuing desire for *human understanding* of a proof, in addition to knowledge that the theorem is true." He goes on to explain,

> People have very powerful facilities for taking in information visually or kinesthetically, and thinking with their spatial sense. On the other hand, they do not have a very good built-in facility for inverse vision, that is, turning an internal spatial understanding back into a two-dimensional image. Consequently, mathematicians usually have fewer and poorer figures in their papers and books than in their heads.

Color versions of one or more of the figures in the article can be found online at www.tandfonline.com/ucmj.
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Luckily, as computers have improved, mathematicians now have more tools available to create useful two- and three-dimensional images that do reflect their internal visualization. Many mathematicians are moving toward a model where they attempt to include their personal visualizations alongside their rigorous proofs. Roger Nelsen's texts exemplify this model, but reverse the roles found in most books—usually his exposition supplements the figures.

A new era of visualization

Roger Nelsen is a professor emeritus at Lewis and Clark College, after an active teaching and research career lasting over 40 years. He has served as a model mathematician over that time, establishing himself as a successful researcher (his *An Introduction to Copulas*, now in its second edition, has over 800 citations on Mathematical Reviews and more than 11,000 citations on Google Scholar), a life-long learner, and a successful teacher.

In 1987, Nelsen published his first Proof Without Words (PWW) in *Mathematics Magazine* and began collecting PWWs he was subsequently asked to review [**8**]. After six years, he used his collection in the first PWW compendium [**13**]. Once you have worked your way through all 104 of the results in PWW volume 1, you can move on the 2000 volume 2 [**14**] and then the 2015 volume 3 [**16**]. These books place Nelsen almost on the opposite side of the spectrum from Bourbaki—here are mathematical works that include almost no words! The geometric pictures and algebraic diagrams provide a playground for mathematics learners to experience all three stages of Tao's view of mathematics education.

The PWW compendia might be too anti-Bourbaki for you; in that case, Nelsen ramped up his exposition in 2006, publishing five books so far with Claudi Alsina [**1**–**3**, **5**, **6**]. While many of these texts overlap content and diagrams (the non-PWW texts almost always refer back to diagrams from the PWW books), they each have different specified goals.

Thurston [**23**] explains that there is "joy [in] discovering new mathematics, rediscovering old mathematics, learning a way of thinking from a person or text, or finding a new way to explain or to view an old mathematical structure." Nelsen's texts align well with this joy by providing visual or clever proofs of facts I have known for years, by introducing me to well-known (but new to me) theorems, and by encouraging me to think about how to use visual proofs with my students and in my own personal research (including finding new visual proofs). Here, I provide select diagrams ranging over the books to demonstrate how useful Nelsen's texts can be for readers of this JOURNAL.

Teaching

When I first heard of Proofs Without Words, I assumed that they were all about geometry, but I was clearly mistaken. All the PWW compendia include sections on geometry and trigonometry, but I avoid most of those uses here, as trigonometry and geometry classes are typically already diagram-heavy. This also means that we will not discuss Alsina and Nelsen's 2015 book on solid geometry [**6**]. Instead, we consider here a variety of geometric diagrams and discuss some of their other uses in several other courses.

Divided squares. The area model for algebraic identities is very powerful and reaches across the entire undergraduate curriculum (at least). Four diagrams that appear quite regularly across Nelsen's texts are shown in Figure 1.

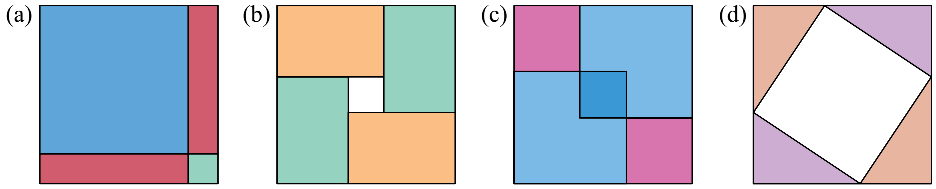

Figure 1. Four algebraic area diagrams built from squares.

The square in Figure 1(a) with side lengths $a + b$ arises naturally in both high school and introductory college algebra; this diagram provides a visual justification for the classic (and oft mistaken) formula for the square of a sum: $(a + b)^2 = a^2 + 2ab + b^2$ for all real numbers a and b. Nelsen uses the same diagram to demonstrate the formula for the square of a sum of two functions [**15**]. And with large square area interpreted as x^2, each rectangle area as $ax/2$, and small square area as $(a/2)^2$, the diagram also models completing the square [**13**], making it suitable for students from precalculus to calculus (when learning integration techniques).

Nelsen calls Figure 1(b) a Tatami diagram [**5**] since it suggests a certain tiling placement of mats. Along with providing one of the classic PWWs for the Pythagorean theorem, the Tatami can also be used to prove that $(a + b)/2 \geq \sqrt{ab}$ for all for all positive real numbers a and b, the arithmetic mean–geometric mean inequality (AM-GM).

The AM-GM inequality may be the most-proved fact throughout Nelsen's books and all PWWs—he has proofs based on each of Figure 1(b)–(d), the semicircle (Figure 4), a right triangle, the rectangular hyperbola (Figure 5), and more. The sheer volume of proofs provided for this one fact might seem like overkill; however, Nelsen and his coauthors utilize the AM-GM inequality so often in a variety of ways that it becomes clear why they offer so many proofs.

Nelsen uses the "overlapping areas" diagram of Figure 1(c) to visualize the irrationality of $\sqrt{2}$; using this idea with students is expanded in the recent Polster and Ross article in this Journal [**18**]. In particular, the diagram gives students the opportunity to visualize the process of infinite descent and see the resulting contradiction, which I believe enhance students' understanding of the role of contradiction in proof.

Once you learn overlapping diagrams, Nelsen's texts suggest a variety of places to employ them including proofs of Chebyshev's and other inequalities, summing cubes, etc. Overlapping diagrams, the Tatami, and generalizations are used to visualize various facts about Fibonacci numbers in almost all of the books. It is easy for a mathematics enthusiast to get lost trying to create visualizations of their favorite Fibonacci and related number identities (perhaps you might even find a new PWW).

The final divided square diagram, Figure 1(d), known as *Zhou bi suan jing* for the ancient Chinese text where it appears, provides another proof of the Pythagorean theorem. However, the result that caught my eye is the use of more general version of this diagram (opposite triangles are still congruent but the two pairs need not match) to prove the Cauchy–Bunyakovsky–Schwarz inequality for two variables [**2**]:

$$|ab + xy| \geq \sqrt{a^2 + b^2}\sqrt{x^2 + y^2}$$

for all real numbers a, b, x, y. Students learning linear algebra could benefit from seeing the geometric intuition behind this algebraic inequality. It is not surprising that Alsina and Nelsen provide at least four proofs of this fact (and the general version), and that one follows from the AM-GM inequality. Moreover, they note that another

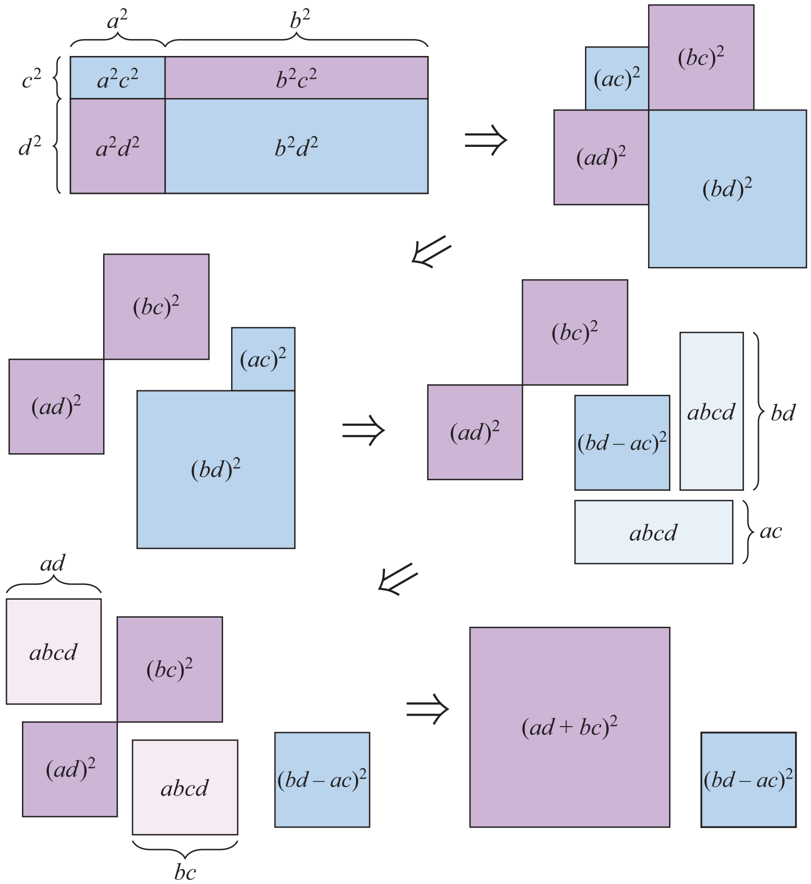

Figure 2. Diophantus's sum of squares identity [13].

proof of the Cauchy–Bunyakovsky–Schwarz inequality follows from another visual algebraic identity, Diophantus's sum of squares identity:

$$(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2$$

for all real numbers a, b, c, d. The visual proof of this identity, shown in Figure 2, also demonstrates the algebra required to show that the modulus of the product of two complex numbers is the product of the moduli, $|z_1z_2| = |z_1||z_2|$ for all $z_1, z_2 \in \mathbb{C}$. In turn, this fact provides a visual proof that the unit circle $T = \{z \in \mathbb{C} \mid |z| = 1\}$ is closed under multiplication. Students in a modern algebra class can thus use this fact, without knowing the polar form of complex numbers, as part of the proof that T is a subgroup of the nonzero complex numbers under multiplication.

I find it truly amazing that the four diagrams of Figure 1 can be used with students ranging from introductory college algebra to calculus, combinatorics, linear algebra, real analysis, and on to modern algebra while also introducing me to mathematical ideas and techniques I had not previously known.

Triangles and semicircles. As mentioned before, the semicircle and the right triangle provide the bases for visual proofs of nearly every imaginable trigonometric identity and rule. In particular, many such visual proofs use Thales's triangle theorem:

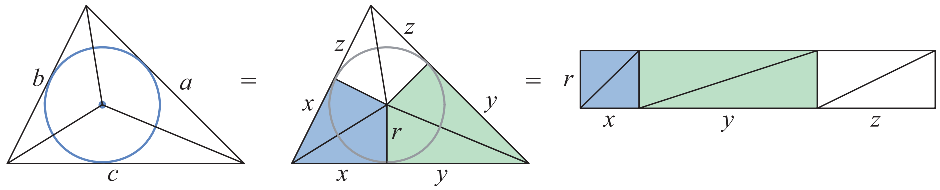

Figure 3. A triangular diagram.

A triangle inscribed in a semicircle is a right triangle; see Figure 4(a). We avoid these classic visualizations of trigonometry truths, but we would be remiss not to discuss how Nelsen uses both triangles and the semicircle to demonstrate other facts in the undergraduate curriculum. For example, the right triangle with appropriate labels can be used to prove the existence of infinitely many primitive Pythagorean triples: For all integers $m > n > 0$ with $\gcd(2mn, m^2 - n^2) = 1$, the three numbers $m^2 - n^2$, $2mn$, and $m^2 + n^2$ form a primitive Pythagorean triple [**14**]. This serves as an excellent exercise for students in geometry and number theory classes alike.

The right triangle can also be used to prove the inequality $\sum_{k=1}^{n} 1/\sqrt{k} > \sqrt{n}$ for $n > 1$ [**14**]. This fact is often used as an exercise for students learning induction, but it can also be readily used in a calculus class to directly prove the divergence of the series $\sum_{k=1}^{\infty} 1/\sqrt{k}$ which typically requires the integral test or comparison test.

The triangular diagram shown in Figure 3 appears many times throughout Nelsen's books. This diagram helped me learn a lot about triangle centers (discussed below), but also provides a surprising visualization of Heron's formula for the area of a triangle: A triangle with side lengths a, b, and c has area $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = (a+b+c)/2$ is the semiperimeter.

The semicircle of Figure 4(a) can be used for yet another proof of the AM-GM inequality. A modified diagram (moving the triangle to have one endpoint in the center) yields proofs of other inequalities such as the geometric mean–harmonic mean inequality [**14**]: $\sqrt{ab} \geq 2ab/(a+b)$ for all positive real numbers a, b.

However, when shaded as in Figure 4(b), the semicircle can provide calculus students visual justification for the integral formula for the circle,

$$\int_a^1 \sqrt{1-x^2}\,dx = \frac{\cos^{-1}(a)}{2} + \frac{(-a)\sqrt{1-a^2}}{2}$$

for any $-1 \leq t \leq 0$; no trigonometric substitution needed! This wonderful use of the semicircle [**16**] is so engaging that it already appears in calculus text exercises [**21**].

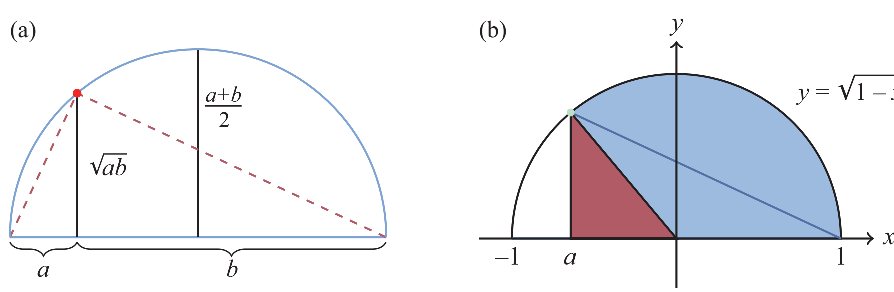

Figure 4. Two applications of the unit semicircle.

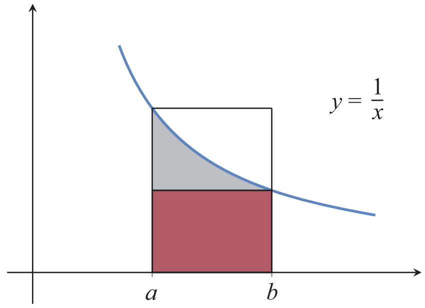

Figure 5. The rectangular hyperbola, labeled to prove Napier's inequality.

Hyperbola. Continuing with calculus, Nelsen and his coauthors have mastered using the graph of $y = 1/x$ to demonstrate a variety of proofs. The simple image in Figure 5, labeled appropriately (often with extra lines or shading), provides visual justification for all of the following statements (and more).

- $\ln(ab) = \ln(a) + \ln(b)$ for all a, b;
- $\ln(1/a) = -\ln(a)$ for all a;
- Napier's inequality: $\dfrac{1}{b} < \dfrac{\ln(b) - \ln(a)}{b - a} < \dfrac{1}{a}$ for all positive real numbers $a < b$;
- logarithmic mean inequality: $\sqrt{ab} < \dfrac{b - a}{\ln(b) - \ln(a)} < \dfrac{a + b}{2}$ for all positive real numbers $a < b$;
- $\sum_{k=1}^{\infty} \dfrac{1}{T_k} = 1 + \dfrac{1}{3} + \dfrac{1}{6} + \dfrac{1}{10} + \cdots = 2$ where $T_k = \dbinom{k}{2} = \dfrac{k(k+1)}{2}$;
- the series $\sum_{k=1}^{\infty} \dfrac{1}{k}$ diverges and the series $\sum_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k}$ converges to $\ln(2)$.

As with all the diagrams so far, I was originally surprised by the mathematical mileage one can get from such a simple image. Students often misunderstand logarithms; by the time they have taken integral calculus, perusing Nelsen's texts and the proofs related to this diagram can be very illuminating.

Various arrays. As someone with interests in combinatorics, number theory, and integer sequences, I especially enjoy the PWWs with arrangements of squares and dots for both myself and sharing with students. In any standard calculus class, discrete mathematics class, or introduction to proofs class (and even real analysis), students typically encounter theorems about algebraic sums of powers, such as

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

These formulas can seem unmotivated and dry to calculus students or those learning proof by induction, but the diagrams in Figure 6 can help students understand them.

Figure 6(a) is a classic visual of the formula for the sum of consecutive integers (giving the formula for the triangular numbers). This diagram reflects the mathematical legend of Gauss summing the first 100 positive integers. Figure 6(b) is my favorite of 12 different diagrams in the books proving the formula for the sum of squares; this diagram and its siblings seem to be less well known.

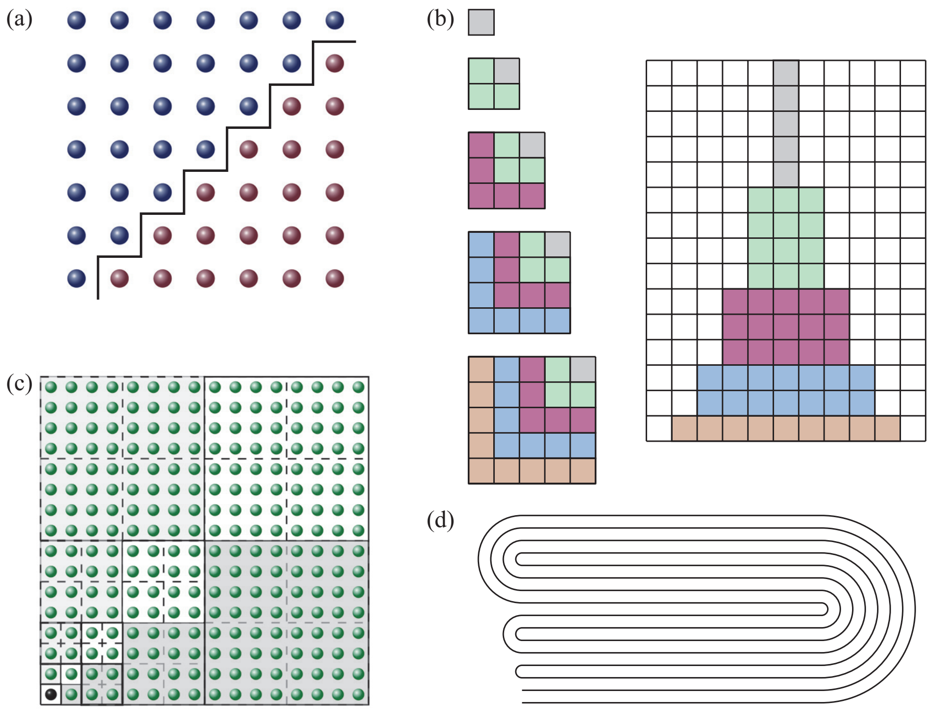

Figure 6. Visualizations for some geometric sums.

We need not stop with algebraic sums. The first two of the geometric sums

$$1+\sum_{i=0}^{n} 2^i = 2^{n+1}, \quad 1+3\sum_{i=0}^{n} 4^i = 4^{n+1}, \quad 1+(b-1)\sum_{i=0}^{n} b^i = b^{n+1}$$

have PWWs shown in Figure 6(d) and (c), respectively, while [16] includes a technique for the general result for all natural numbers b. Figure 7 shows two self-similar diagrams for the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1, \quad \sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{1}{3}.$$

Arrays of triangles and boxes allow students to discover number theoretic statements as well. For instance, Figure 8 demonstrates that the squares can be visualized

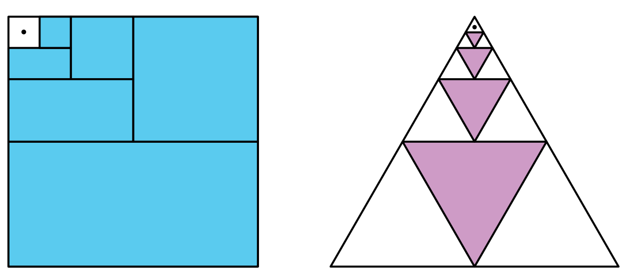

Figure 7. Self-similar diagrams.

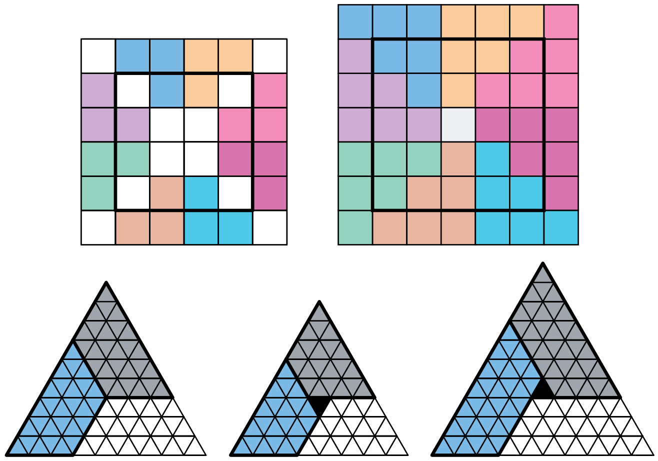

Figure 8. Squares and triangles representing squares.

as square arrays or in a triangular array of triangles; these help prove that, for every natural number n,

$$n^2 \equiv 0 \text{ or } 1 \bmod 3, \quad n^2 \equiv 0 \text{ or } 1 \bmod 4.$$

Visualizing the squares as triangular arrays of triangles also provides explanations of why the sum of the first n odd natural numbers is n^2 (as illustrated on the cover of this issue), demonstrates the sum of squares formula and various Fibonacci identities such as

$$1 + \sum_{k=1}^{n} F_k = F_{n+1}, \quad F_n^2 + F_{n+1}^2 + \sum_{k=1}^{n} 2F_k^2 = (F_n + F_{n+1})^2$$

where $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ [16]. For many more related results, Nelsen's new *Nuggets of Number Theory* [17] will be published soon after this review.

In the classroom, Proofs Without Words can provide additional means for understanding results. Moreover, some PWW diagrams such as those in Figure 6 can allow students to arrive at the associated formulas themselves: Calculus students can guess the formulas and make a visual connection while students in more advanced classes (such as discrete mathematics, introduction to proofs, and number theory) can use the diagrams to determine and state the theorem and also uncover a proof (whether by induction or a combinatorial proof). The "inverse problem" of determining the result given the diagram can be an engaging exercise.

Learning

My initial foray into Proofs Without Words came as a learner. As an avid reader of both *The College Mathematics Journal* and *Mathematics Magazine*, I became intrigued by the various PWWs (since I often did not immediately understand them) and sought out the first two compendia. As I became more familiar with the techniques, I began to see connections between diagrams, which became clearer once I started reading Nelsen's other texts. Then I really began to learn new mathematical techniques and theorems.

For instance, Figure 1(c) introduced me to the carpets theorem when I learned about the visual proof that $\sqrt{2}$ is irrational [5]: For two carpets on the floor of a room, the area of the overlap equals the uncovered area if and only if the combined area of the carpets equals the area of the floor. One of the standard proofs of Fermat's theorem on the sum of two squares makes heavy use of Diophantus's identity of Figure 2. Nelsen uses Figure 3 to introduce a technique known as Ravi substitution, where a, b, c are written in terms of x, y, z. This technique has become popular in contest problems and helps with proofs of many inequalities, including Padoa's: For a, b, c the sides of a triangle, $abc \geq (a+b-c)(b+c-a)(c+a-b)$.

Further manipulation of Figure 3 also introduced me to the fascinating study of circle centers and the inradius of a triangle, including a famous relationship between the circumradius and inradius of a triangle due to Euler: The product of the perimeter of a triangle and its inradius is twice the area of the triangle, and the circumcenter of a triangle is greater than or equal to twice the inradius of the triangle. Nelsen's books skillfully guide you through the varied uses of this one diagram to show how the many results are interconnected. Moreover, Figure 3 led me to learn about various triangle centers and to think carefully about the Fermat point (the "center" of the triangle minimizing the sum of distances to the vertices), which has in turn led to an interesting project for undergraduates. Nelsen's expositions and emphasis on visual diagrams tend to encourage me to investigate topics that might otherwise be outside my comfort zone.

As another example, even though I knew about the Euler–Mascheroni constant

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln(n+1)\right),$$

visualizing its existence led me to think about other constants as areas [3, 15]. Every time I open one of Nelsen's books, I find new-to-me mathematics or discover an interesting problem to consider. Moreover, *Math Made Visual* [1] has encouraged me learn more about visualizing problems to help me be a better educator and mathematician.

Discovery

As we learn more, we find new ways to contribute to the existing body of knowledge. Nelsen's texts have inspired me to think carefully about visualization not only in my teaching and learning processes, but also in my personal research. While my personal visualizations might not be as clean or clear as the beautiful diagrams populating Nelsen's books, his texts have influenced me to try to refine the pictures in my head to include in my papers. For example, my note [11] stemmed from trying to understand Cardano's and Viéte's solution to the cubic equation in a Proof Without Words; it turned out to exceed the tight bounds of the PWW format (for me), but the visuals helped me, and hopefully others, understand the transformations involved in that algebraic solution.

My graduate advisor would explain his thought process about certain theorems by describing a visual he had in his head of traversing a partially ordered set; however, none of that visualization was included in his papers. If he had managed to include such imagery, would it help others better understand the results? I believe so, and I would further suggest that the mathematics community would be better off if we found a way to turn our "internal spatial understanding back into a two-dimensional image," as discussed by Thurston. While learning from Nelsen's visualization techniques, you might be inspired to think about using visuals in or stemming from your own work. As an example of this, Brian Hopkins explains in a recent PWW [**12**] how his research in Bulgarian Solitaire inspired a diagram showing a result about products of squares and triangular numbers; I had not previously heard of Bulgarian Solitaire and the visual has helped me keep the idea in my head to discuss the topic with others.

Alsina and Nelsen's article "Invitation to Proofs Without Words" [**4**] and book *Math Made Visual* [**1**] have inspired me to attempt visual proofs of my own, requiring me to think carefully about which of my favorite theorems can be proved in this way. For example, I often work with undergraduates investigating combinatorial ideas related to positional numeration systems. Figure 6(c,d) implies the existence of the base-2 and base-4 representations of numbers and inspired me to create my own PWW about factorial sums implying the existence of the factorial base number system [**10**]. That diagram was very useful during a 2017 Research Experience for Undergraduates for describing some combinatorial properties of the factorial base positional system. Without the pictorial representation, we might not have uncovered our results.

As another example, using the rectangular hyperbola to show the convergence of the alternating harmonic series inspired my coauthor and me to create our own decomposition diagram demonstrating the convergence of a rearrangement of the alternating harmonic series [**7**], providing a visual way to introduce the counterintuitive Riemann rearrangement theorem. Furthermore, when I learned about Figure 7, I tried to generalize to similar triangular diagrams, resulting in [**9**], which led me to an interesting theorem about decomposing triangles into smaller congruent triangles. Eventually, I discovered a fascinating open problem: Do there exist infinitely many positive integers n for which $\gcd(2^n - 1, 3^n - 1) = 1$? I have learned that following a diagram inevitably leads to an interesting question or new-to-me mathematical ideas.

Finally, I leave the reader with two challenges that I have undertaken since first encountering Nelsen's books. First, find a visual proof (via a decomposition of a shape with area 1) of the Goldbach–Euler theorem, that $\sum_p 1/(p-1) = 1$ where p ranges over the set of distinct perfect powers (except 1). Second, think of your favorite classical theorem (or your own result); can you give a visual proof?

Conclusion

Nelsen and his coauthors pack their texts with examples ranging over the undergraduate curriculum to share with students and colleagues. Of course there is heavy emphasis on topics in the calculus sequence, combinatorial number theory, and trigonometry and geometry, but if you spend time digesting the ideas in Nelsen's texts, then you will begin to see how to utilize diagrams in other classes; you might even see how to modify existing diagrams to suit your needs for other results. If this is your goal, I recommend starting with *Icons of Mathematics* [**5**] or *Math Made Visual* [**1**]. If you want to know more about visualization in calculus, I recommend starting with *Cameos for Calculus* [**15**] which can lead to Nelsen's other calculus texts. For a broad understanding of the general Proofs Without Words landscape, I suggest starting with the three compendia [**13**, **14**, **16**] and following your interests to the other books; the expository texts are all rooted in PWW diagrams.

I imagine any reader of this JOURNAL is familiar with Proofs Without Words, with new examples included in almost every issue, but they might not be aware of the interconnectedness of the various diagrams. Nelsen tells many historical and mathematical stories through these diagrams and provides several examples we can use as teachers, as learners, and even as explorers of new mathematical ideas.

I recommend that every university library (or every mathematician) have at least one of Nelsen's books. Check out any (or all) of them and I imagine that it will not take too long for you to become intrigued by his diagrams and clever proofs of classical results. Eventually, you might want to see if you can prove your favorite results just as elegantly; you might even advance your own research in the process. Perhaps it could become the norm to include a diagram alongside logical proofs, formally merging the Bourbaki and anti-Bourbaki communities to encourage intuition that is rooted in rigorous foundations.

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