THE DISTRIBUTION OF THE NUMBER OF PARTS OF $m$-ARY PARTITIONS MODULO $m$

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ABSTRACT. We investigate the number of parts modulo $m$ of $m$-ary partitions of a positive integer $n$. We prove that the number of parts is equidistributed modulo $m$ on a special subset of $m$-ary partitions. As consequences, we explain when the number of parts is equidistributed modulo $m$ on the entire set of partitions, and we provide an alternate proof of a recent result of Andrews, Fraenkel and Sellers regarding the number of $m$-ary partitions modulo $m$.

1. Preliminaries and statement of the main result. Throughout this note, we let $\mathbb{N} = \{0, 1, 2, 3, \ldots \}$ represent the set of natural numbers. For any $m \geq 2$, every natural number $n$ has a unique base-$m$ representation of the form $n = n_0 + n_1 m + \cdots + n_k m^k$ with $n_k \neq 0$. We express this more compactly as $n = (n_0, n_1, \ldots, n_k)_m$ and use the convention that $n_i = 0$ if $i > k$.

For $m \geq 2$, we say that a partition of $n \in \mathbb{N}$ is an $m$-ary partition if each part is a power of $m$. We let $b_m(n)$ represent the number of $m$-ary partitions of $n$. For instance, the 2-ary partitions of 8 are

\[
8, \quad 4 + 4, \quad 4 + 2 + 2, \quad 4 + 2 + 1 + 1, \\
4 + 1 + 1 + 1 + 1, \quad 2 + 2 + 2 + 2, \\
2 + 2 + 2 + 1 + 1, \quad 2 + 2 + 1 + 1 + 1 + 1, \\
2 + 1 + 1 + 1 + 1 + 1 + 1 + 1, \quad 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1,
\]

such that $b_2(8) = 10$.

In a recent article, Andrews, Fraenkel and Seller, see [3], provided the following beautiful characterization of the number of $m$-ary partitions mod $m$ relying only on the base-$m$ representation of a number.

2010 AMS Mathematics subject classification. Primary 05A17, 11P83.
Keywords and phrases. Partitions, $m$-ary partitions, congruence properties.
Received by the editors on January 20, 2016.

DOI:10.1216/RMJ-2017-47-6-1 Copyright ©2017 Rocky Mountain Mathematics Consortium
**Theorem 1.1 ([3]).** If $m \geq 2$ and $n = (n_0, n_1, \ldots, n_k)_m$, then

$$b_m(mn) = \prod_{i=0}^{k} (n_i + 1) \mod m.$$ 

Their elegant proof follows from clever manipulation of power series and the generating function for $m$-ary partitions. Their result allows for a uniform proof of many known congruence properties of $m$-ary partitions, originally conjectured by Churchhouse and proved by Redseth, Andrews and Gupta, see [1, 6, 8, 9, 10].

Theorem 1.1 implies that

$$b_m(mn) - \prod_{i=0}^{k} (n_i + 1) = m \cdot q$$

for some $q \in \mathbb{N}$. Our primary result (Theorem 1.2) provides a combinatorial interpretation for the value of $q$. Furthermore, as a corollary to our main result, we obtain a new proof of Theorem 1.1 which does not rely on generating functions.

Note that the product in Theorem 1.1,

$$\prod_{i=0}^{k} (n_i + 1),$$

arises in various other places; for instance, when $m$ is prime, this number counts the nonzero entries in row $n$ of Pascal’s triangle mod $m$, see [7]. This product may also be interpreted in terms of a partial order on the natural numbers arising from base-$m$ representations. In particular, for fixed $m \geq 2$, we let $\preceq_m$ represent the $m$-dominance order defined by $a \preceq_m b$ if $a_i \leq b_i$ for all $i$, where $a = (a_0, a_1, \ldots, a_k)_m$ and $b = (b_0, b_1, \ldots, b_l)_m$, see [4, 5]. Then, for $n = (n_0, n_1, \ldots, n_k)_m$, the same product counts the number of integers dominated by $n$, see [4]. We will use the interpretation of the product in terms of the $m$-dominance order in what follows.

Now, let $n$ be a positive integer with $m^k \leq n < m^{k+1}$. Then, every $m$-ary partition is of the form

$$\ell_k \cdot m^k + \ell_{k-1} \cdot m^{k-1} + \cdots + \ell_1 \cdot m + \ell_0.$$
with \( \ell_i \geq 0 \) for all \( i \). We will denote such a partition by \([\ell_0, \ell_1, \ldots, \ell_{k-1}, \ell_k]_m\). It is noteworthy to mention here that the base-\( m \) representation of \( n \) yields an \( m \)-ary partition

\[
(n_0, n_1, \ldots, n_k)_m \mapsto [n_0, n_1, \ldots, n_k]_m.
\]

Finally, we define a function \( \text{nops} \) from \( m \)-ary partitions of \( n \) to \( \mathbb{N} \) by

\[
\text{nops}([\ell_0, \ell_1, \ldots, \ell_{k-1}, \ell_k]_m) = \sum_{i=0}^{k} \ell_i;
\]

this represents the number of parts of the partition.

Now, let \( n = (n_0, n_1, \ldots, n_k)_m \). We call an \( m \)-ary partition, \( \ell \), of \( n \) simple if \( \ell = [\ell_0, \ell_1, \ldots, \ell_k]_m \) with \( \ell_i \leq n_i \) for all \( i \geq 1 \). Thus, simple partitions are obtained by replacing powers of \( m \) in the \( m \)-ary representation with the appropriate number of 1s. Let \( P_m(n) \) be the set of \( m \)-ary partitions of \( n \), \( S_m(n) \) the set of simple \( m \)-ary partitions of \( n \) and \( N_m(n) = P_m(n) \setminus S_m(n) \) the set of non-simple \( m \)-ary partitions of \( n \). Restricting the function \( \text{nops} \) to \( N_m(n) \), we obtain the following result.

**Theorem 1.2.** Let \( m \geq 2 \) and \( n \in \mathbb{N} \). Then, the \( \text{nops} \) function is equidistributed modulo \( m \) on the set \( N_m(n) \).

As a corollary, we obtain the following.

**Corollary 1.3.** Let \( b_m(n) \) be the number of \( m \)-ary partitions of \( n = (n_0, n_1, \ldots, n_k)_m \). Then

\[
b_m(n) \equiv \prod_{i=1}^{k} (n_i + 1) \mod m.
\]

Note that the previous corollary is stated slightly differently than Theorem 1.1, which is given only for \( b_m(mn) \); however, due to the fact that \( b_m(mn+r) = b_m(mn) \) when \( 0 < r < m \) (as stated in [3]), the two forms are equivalent.

This paper is organized as follows. Section 2 contains the details necessary to prove Theorem 1.2. We prove the theorem and its corollary
in Section 3. In addition, we use Theorem 1.2 to prove that the \textit{nops} function is equidistributed mod \( m \) on the entire set of \( m \)-ary partitions, \( P_m(n) \), if and only if \( m - 1 \) appears in the base-\( m \) representation of \( n \), see Theorem 3.2. Section 4 contains a detailed example illustrating the results in Sections 2 and 3. Finally, in Section 5, we describe some possible extensions.

2. Technical details. In this section, we provide a systematic method for partitioning \( N_m(n) \), which will be used to prove Theorem 1.2. We have included a detailed example of this method of partitioning in Section 4.

Let \( m \geq 2 \) and \( n \in \mathbb{N} \) be fixed with \( n = (n_0, n_1, \ldots, n_k)_m \). First, we define a function \( f_{m,n} : N_m(n) \to \mathbb{N} \) by

\[
f_{m,n}([\ell_0, \ell_1, \ldots, \ell_k]_m) = (b_0, b_1, b_2, \ldots, b_k)_m,
\]

where \( b_i = \min(n_i, \ell_i) \) for all \( i \); note that \( b_0 = n_0 \) since \( \ell_0 \equiv n_0 \) (mod \( m \)). The next lemma follows by construction.

**Lemma 2.1.** For any non-simple partition \( \ell \in N_m(n) \), we have \( f_{m,n}(\ell) \ll_m n \).

Now, we use \( f_{m,n} \) to define a relation on \( N_m(n) \) by \( \rho \sim \gamma \) if \( f_{m,n}(\rho) = f_{m,n}(\gamma) \).

**Lemma 2.2.** The relation \( \sim \) is an equivalence relation, and thus,

\[
\{ f_{m,n}^{-1}(b) \mid b \in \mathbb{N} \text{ and } b \ll_m n \text{ and } f_{m,n}^{-1}(b) \neq \emptyset \}
\]

forms a partition of \( N_m(n) \).

**Proof.** Any function yields such an equivalence relation. \( \square \)

**Lemma 2.3.** Let \( \ell \) be a non-simple \( m \)-ary partition of \( n \). Then \( \ell \) can be component-wise decomposed as

\[
\ell = [\ell_0, \ell_1, \ldots, \ell_k]_m = [r_0, r_1, \ldots, r_k]_m + [b_0, b_1, b_2, \ldots, b_k]_m,
\]

where \( b = (b_0, b_1, b_2, \ldots, b_k)_m = f_{m,n}(\ell) \) and \( r_i \geq 0 \) for all \( i \). Moreover, it follows that \( r_i > 0 \) only if \( n_i = b_i \).
Proof. Since \( r_i = \ell_i - \min(\ell_i, n_i) \), it is clear that \( r_i \geq 0 \). Now, if \( r_i > 0 \), then \( \min(\ell_i, n_i) \neq \ell_i \) so that \( b_i = n_i \), as required. 

Lemma 2.4. Let \( \ell \) be a non-simple \( m \)-ary partition of \( n = (n_0, n_1, \ldots, n_k)_m \) with \( \ell \in \mathcal{F}_{m,n}^{-1}(b) \) where \( b = (b_0, b_1, \ldots, b_k)_m \). Suppose that \( \ell \) is of the form

\[
\ell = [\ell_0, b_1, b_2, \ldots, b_{j-1}, \ell_j, \ell_{j+1}, \ldots, \ell_k]_m
\]

with \( \ell_j > n_j = b_j \). Then, there is a unique pair \((r, h)\) with \( r \geq 1 \) and \( 0 \leq h < m^j \) such that \( \ell_j \leq n_j + mr \), there is an \( m \)-ary partition of the form \([h, b_1, b_2, \ldots, b_{j-1}, b_j + mr, \ell_{j+1}, \ldots, \ell_k]_m \), and there is no \( m \)-ary partition of the form \([h', b_1, b_2, \ldots, b_{j-1}, g, \ell_{j+1}, \ldots, \ell_k]_m \) with \( g > b_j + mr \).

Proof. Let \( s = \ell_j - b_j = \ell_j - n_j > 0 \). According to the division algorithm, there is a unique \( h \) satisfying \( \ell_0 = t \cdot m^j + h \) where \( 0 \leq h < m^j \). Then, clearly,

\[
[h, b_1, b_2, \ldots, b_{j-1}, b_j + s + t, \ell_{j+1}, \ldots, \ell_k]_m
\]

is an \( m \)-ary partition of \( n \). Note that

\[
[h', 0, 0, \ldots, 0, b_j + s + t, \ell_{j+1}, \ldots, \ell_k]_m
\]

is an \( m \)-ary partition of \( n \) where

\[
h' := h + \sum_{i=1}^{j-1} b_i = \sum_{i=0}^{j-1} n_i < m^j.
\]

This implies that

\[
[0, 0, 0, \ldots, 0, b_j + s + t, \ell_{j+1}, \ldots, \ell_k]_m
\]

is an \( m \)-ary partition of \( n' = (0, 0, \ldots, 0, n_j, n_{j+1}, \ldots, n_k)_m \). However, since \( n_j = b_j \) and \( s + t > 0 \), then

\[
0 < s + t = \sum_{i=j+1}^{k} (n_i - \ell_i) \cdot m^{i-j}.
\]

Thus, \( s + t = mr \) for some \( r \geq 1 \), as required. Finally, we see that \( b_j + mr \) is the largest number of parts of the form \( m^j \) we can have without reducing some \( \ell_i \) with \( i > j \). 

Corollary 2.5. Let \( \ell \) be a non-simple \( m \)-ary partition of \( n = (n_0, n_1, \ldots, n_k)_m \) with \( \ell \in f_{m,n}^{-1}(b) \) where \( b = (b_0, b_1, \ldots, b_k)_m \). Suppose that \( \ell \) is of the form

\[
\ell = [\ell_0, b_1, b_2, \ldots, b_{j-1}, \ell_j, \ell_{j+1}, \ldots, \ell_k]_m
\]

with \( \ell_j > n_j = b_j \). Then, there is an \( m \)-ary partition of the form

\[
[v, b_1, b_2, \ldots, b_{j-1}, u, \ell_{j+1}, \ldots, \ell_k]_m
\]

for all \( b_j < u \leq b_j + mr \) where \( r \) is given by Lemma 2.4.

Proof. Let \( b_j < u \leq b_j + mr \), and consider the partition of the form

\[
\rho = [h, b_1, b_2, \ldots, b_{j-1}, b_j + mr, \ell_{j+1}, \ldots, \ell_k]_m
\]

guaranteed by Lemma 2.4. Then we find \( y \) such that

\[
b_j + mr = u + y \text{ where } y \geq 0.
\]

Next, construct an \( m \)-ary partition from \( \rho \) by converting \( y \) parts of the form \( m^0 \) to \( y \cdot m^j \) parts of the form \( m^0 \), obtaining the partition

\[
[h + y \cdot m^j, b_1, b_2, \ldots, b_{j-1}, u, \ell_{j+1}, \ldots, \ell_k]_m,
\]

as required. \( \square \)

Now, fix \( b \ll_m n \) with \( f_{m,n}^{-1}(b) \neq \emptyset \). For each \( 1 \leq z \leq k \), we define

\[
B(z) := \{ \rho \in f_{m,n}^{-1}(b) \mid \min\{i \geq 1 \mid \rho_i \neq b_i\} = z \}.
\]

Again, the following lemma is clear by construction.

Lemma 2.6. Let \( b \ll_m n \) with \( f_{m,n}^{-1}(b) \neq \emptyset \). Then, the collection of sets \( \{B(z) \mid B(z) \neq \emptyset\} \) forms a partition of \( f_{m,n}^{-1}(b) \).

As our final step, we fix \( z \) with \( 1 \leq z \leq k \) such that \( B(z) \neq \emptyset \). Now, we define a relation on \( B(z) \) as follows. We say that \( \rho \simeq_{b,z} \gamma \) if \( \gamma_i = \rho_i \) for all \( i > z \).

Lemma 2.7. The relation \( \simeq_{b,z} \) on \( B(z) \) is an equivalence relation and thus provides a partition of \( B(z) \).

Proof. This is again clear by construction. \( \square \)

Proposition 2.8. Let \( n \in \mathbb{N} \), \( b \in \mathbb{N} \) with \( b \ll_m n \) and \( 1 \leq z \leq k \) be such that \( f_{m,n}^{-1}(b) \neq \emptyset \) and \( B(z) \neq \emptyset \). Then, the nops function is equidistributed modulo \( m \) on each equivalence class of \( \simeq_{b,z} \).
Proof. Suppose that $C$ is an equivalence class of $\simeq_{b,z}$. Then, by construction, there exists an $\ell_{z+1}, \ell_{z+2}, \ldots, \ell_k$ such that every partition in $C$ is of the form

$$[h, b_1, b_2, \ldots, b_{z-1}, h', \ell_{z+1}, \ell_{z+2}, \ldots, \ell_k]_m$$

for some $h$ and $h'$ with $h' > b_z$. Now, according to Lemma 2.5 and Corollary 2.5, there exists some $r \geq 1$ such that

$$C = \{[h, b_1, b_2, \ldots, b_{z-1}, u, \ell_{z+1}, \ell_{z+2}, \ldots, \ell_k]_m \mid h \in \mathbb{N} \text{ and } b_j < u \leq b_j + mr\}.$$

Thus, $|C| = mr$. Now, for each $1 \leq w \leq m$, we define

$$C_w = \{[h_j, b_1, b_2, \ldots, b_j + w + jm, \ell_{z+1}, \ell_{z+2}, \ldots, \ell_k]_m \mid 1 \leq j \leq (r-1)\},$$

and we note that $|C_w| = r - 1$ for all $w$ and the set $\{C_w\}$ forms a partition of $C$. Moreover, for each $w$, $\nops(\gamma) \equiv \nops(\rho) \pmod{m}$ for all $\gamma, \rho \in C_w$, and $\nops(\rho) \equiv \nops(\gamma) + 1 \pmod{m}$ whenever $\gamma \in C_w$ and $\rho \in C_{w+1}$.

3. Proof of Theorem 1.2 and consequences.

Proof of Theorem 1.2. Let $b \ll n$ with $f_{m,n}^{-1}(b) \neq \emptyset$. Then, let $1 \leq z \leq k$ with $B(z)$ be non-empty. By Proposition 2.8 and Lemma 2.7, the $\nops$ function is equidistributed mod $m$ on $B(z)$. Likewise, by Lemma 2.6, the $\nops$ function is equidistributed mod $m$ on $f_{m,n}^{-1}(b)$. Finally, Lemma 2.2 implies that the $\nops$ function is equidistributed mod $m$ on $N_m(n)$.

Let $n = (n_0, n_1, \ldots, n_k)_m$. Then, according to Theorem 1.2, $N_m(n) = m \cdot q$, where $q$ is the number of non-simple $m$-ary partitions with the number of parts divisible by $m$. However, it is clear that there is a bijection between simple $m$-ary partitions of $n$ and the integers equivalent to $n \mod m$ that are $m$-dominated by $n$:

$$[\ell_0, b_1, b_2, \ldots, b_k]_m \leftrightarrow (n_0, b_1, b_2, \ldots, b_k)_m.$$

As previously mentioned, there are $\prod_{i=1}^k (n_i + 1)$ integers equivalent to $n \mod m$ that are $m$-dominated by $n$ (see [4] and use the fact that $b$ is...
equivalent to \( n \mod m \) if and only if \( b_0 = n_0 \). Thus, we see that

\[
b_m(n) = |N_m(n)| + |S_m(n)| = m \cdot q + \prod_{i=1}^{k}(n_i + 1);
\]

therefore, Corollary 1.3 holds.

Understanding the \( \text{nops} \) function on \( N_m(n) \) allows us to characterize when the \( \text{nops} \) function is equidistributed \( \mod m \) on the entire set of \( m \)-ary partitions, \( P_m(n) \).

**Corollary 3.1.** The \( \text{nops} \) function is equidistributed modulo \( m \) on \( P_m(n) \) if and only if \( \text{nops} \) is equidistributed modulo \( m \) on the simple \( m \)-ary partitions, \( S_m(n) \).

**Proof.** This follows from Theorem 1.2 since \( P_m(n) \) is the disjoint union of \( N_m(n) \) and \( S_m(n) \). ∎

**Theorem 3.2.** Let \( m \geq 2 \), and let \( n = (n_0, n_1, \ldots, n_k)_m \) be the base-\( m \) representation of \( n \). Then, the \( \text{nops} \) function is equidistributed modulo \( m \) on \( P_m(n) \) if and only if the set \( \{n_1, n_2, \ldots, n_k\} \) contains \( m - 1 \).

**Proof.** First, suppose that \( n_i = m - 1 \) for some \( i \geq 1 \). Due to Corollary 3.1, we need to show that the \( \text{nops} \) function is equidistributed on \( S_m(n) \). Now, for each \( w \in \{0, 1, \ldots, m - 1\} \), let

\[
A_w = \{ \ell \in S_m(n) \mid \ell_i = w \}.
\]

Then, it is clear that \( \{A_w \mid w \in \{0, 1, \ldots, m - 1\}\} \) forms a set partition of \( S_m(n) \). Furthermore, since all the \( m \)-ary partitions in \( A_w \) are simple, there is a bijection \( g_w : A_w \to A_{w'} \) given by

\[
g_w(w)((\ell_0, \ell_1, \ldots, \ell_i, \ldots, \ell_k)) := (\ell_0 + (w - w') \cdot m^i, \ell_1, \ldots, w', \ldots, \ell_k)
\]

such that \( |A_w| = |A_{w'}| \) for all \( w, w' \in \{0, 1, \ldots, m - 1\} \). Finally, let \( \ell \in A_0 \). Then, for each \( w \in \{0, 1, \ldots, m - 1\} \), we have \( \text{nops}(g_w(\ell)) = \text{nops}(\ell) + w \mod m \). Thus, the \( \text{nops} \) function is equidistributed \( \mod m \) on \( S_m(n) \).

Conversely, suppose that \( m - 1 \notin \{n_1, \ldots, n_k\} \). First, assume that the only nonzero base-\( m \) digits are \( n_0 \) and \( n_k \) so that, by assumption,
Then, there are only $n_k + 1 \leq m - 1$ simple partitions, and thus, the $nops$ function cannot be equidistributed modulo $m$ on $S_m(n)$. Next, assume that $0 < n_j \leq m - 2$ for some $1 \leq j < k$. Similar to the previous paragraph, for each $w \in \{0, 1, \ldots, n_j\}$, let

$$A_w = \{ \ell \in S_m(n) \mid \ell_j = w \}.$$  

As before, $|A_w| = |A_{w'}|$ for all $w, w' \in \{0, 1, \ldots, n_j\}$ and, for each $\ell \in A_0$ and each $w \in \{0, 1, \ldots, n_j\}$, we have $nops(g_{0,w}(\ell)) \equiv nops(\ell) + w \pmod m$. Since $n_j \leq m - 2$, the $nops$ function will be equidistributed modulo $m$ on $A_0$ if and only if the $nops$ function is equidistributed modulo $m$ on $A_0$. However, we see that there is a bijection $h : A_0 \to S_m(n - n_j \cdot m^j)$ given by

$$h((\ell_0, \ell_1, \ldots, 0, \ldots, \ell_k)) := (\ell_0 - n_j \cdot m^j, \ell_1, \ldots, 0, \ldots, \ell_k).$$

Moreover, we note that $nops(h(\ell)) \equiv nops(\ell) \pmod m$ such that $nops$ is equidistributed modulo $m$ on $A_0$ if and only if $nops$ is equidistributed modulo $m$ on $S_m(n - n_j \cdot m^j)$, which implies that $nops$ is equidistributed modulo $m$ on $S_m(n)$ if and only if $nops$ is equidistributed modulo $m$ on $S_m(n - n_j \cdot m^j)$. Since the digit sets of $n$ and $n - n_j \cdot m^j$ are identical except in position $j$, we can use this argument to deduce that $nops$ is equidistributed modulo $m$ on $S_m(n)$ if and only if $nops$ is equidistributed modulo $m$ on $S_m(n - \sum_{i=1}^{k-1} n_i \cdot m^i)$. However,

$$n - \sum_{i=1}^{k-1} n_i \cdot m^i = (n_0, 0, \ldots, 0, n_k)$$

and $n_k \leq m - 2$; in this case, we have already shown that $nops$ is not equidistributed modulo $m$ on $S_m(n - \sum_{i=1}^{k-1} n_i \cdot m^i)$. The result follows. 

4. Detailed example. We illustrate the results of the previous two sections with an example. Let $m = 3$, and consider $n = 60 = (0, 2, 0, 2)_3$. Then, the total number of 3-ary partitions of 60 is 117, i.e., $b_3(60) = 117$. Of these 117, there are 9 simple partitions listed in Figure 1.

In Figures 2–7, we list the remaining 108 non-simple partitions, those in $N_3(60)$, using the results in Section 2. The numbers 3-dominated by 60 are

$$0, 3, 6, 27, 30, 33, 54, 57, 60.$$
Let $f$ represent $f_{3,60}$. It turns out that $f^{-1}(54)$, $f^{-1}(57)$ and $f^{-1}(60)$ are all empty. There are 6 partitions in $f^{-1}(0)$ and $f^{-1}(3)$; there are 69 partitions in $f^{-1}(6)$; there are 3 partitions in $f^{-1}(27)$ and $f^{-1}(30)$; and there are 21 partitions in $f^{-1}(33)$. All of the nonempty inverse images are listed in Figures 2–7. The subsets correspond to the nonempty sets $B(z)$ for $1 \leq z \leq 3$, and then, the subsets of $B(z)$ correspond to the partition given by $\simeq_{b,z}$ guaranteed by Lemma 2.7. The most representative example is that of $f^{-1}(6)$ as it contains both $B(1)$ and $B(2)$ ($B(3) = \emptyset$) and $B(1)$ is further partitioned into six equivalence classes for $\simeq_{6,1}$.

We can then check that the cardinality of each of the equivalence classes of $\simeq_{b,z}$ is a multiple of 3, and the $nops$ function is equidistributed mod 3 on these smallest parts (see the proof of Theorem 1.2), thus showing that the $nops$ function is equidistributed on $N_3(60)$. 

**Figure 1.**

**Figure 2.**
Figure 3. $f^{-1}(3); 3 = (0, 1, 0, 0)_3$

Figure 4. $f^{-1}(27); 27 = (0, 0, 0, 1)_3$

Figure 5. $f^{-1}(30); 30 = (0, 1, 0, 1)_3$
5. Extensions. In this section, we briefly discuss a possible way of extending these results to other congruence relations. We note that the set of non-simple \( m \)-ary partitions \( N_m(n) \) can be defined as

\[
N_m(n) = \{ \ell \in P_m(n) \mid \ell_j > n_j \text{ for some } j \geq 1 \},
\]

where \( n = (n_0, \ldots, n_k)_m \) is the base-\( m \) representation of \( n \). Consider the following generalizations. For any \( c \geq 1 \), we let

\[
N_{m,c} = \{ \ell \in P_m(n) \mid \ell_j > n_j, \ell_{j+1} = n_{j+1}, \ldots, \ell_{j+c} = n_{j+c} \text{ for some } j \geq 1 \},
\]
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Figure 7.
where we note that $N_m(n)$ can be interpreted as $N_{m,0}$. Then, we prove an analogous result to Lemma 2.4 that shows $|N_{m,c}| \equiv 0 \pmod{m^{c+1}}$. Therefore, if we can determine the size of the set

$$S_{m,c}(n) := P_m(n) \setminus N_{m,c}(n)$$

using mere knowledge of $n$ (possibly the base-$m$ representation of $n$), then we obtain interesting congruence properties for $b_m(n) \pmod{m^{c+1}}$ for any $c$.

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