ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 47, Number 6, 2017

THE DISTRIBUTION OF THE NUMBER OF PARTS OF m-ARY PARTITIONS MODULO m

TOM EDGAR

ABSTRACT. We investigate the number of parts modulo m of m-ary partitions of a positive integer n. We prove that the number of parts is equidistributed modulo m on a special subset of m-ary partitions. As consequences, we explain when the number of parts is equidistributed modulo m on the entire set of partitions, and we provide an alternate proof of a recent result of Andrews, Fraenkel and Sellers regarding the number of m-ary partitions modulo m.

1. Preliminaries and statement of the main result. Throughout this note, we let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ represent the set of natural numbers. For any $m \geq 2$, every natural number n has a unique *base-m representation* of the form $n = n_0 + n_1 m + \cdots + n_k m^k$ with $n_k \neq 0$. We express this more compactly as $n = (n_0, n_1, \ldots, n_k)_m$ and use the convention that $n_i = 0$ if $i > k$.

For $m \geq 2$, we say that a partition of $n \in \mathbb{N}$ is an *m-ary partition* if each part is a power of m. We let $b_m(n)$ represent the number of m-ary partitions of n. For instance, the 2-ary partitions of 8 are

$$8, \quad 4+4, \quad 4+2+2, \quad 4+2+1+1,$$
$$4+1+1+1+1, \quad 2+2+2+2,$$
$$2+2+2+1+1, \quad 2+2+1+1+1+1,$$
$$2+1+1+1+1+1+1,$$
$$1+1+1+1+1+1+1+1,$$

such that $b_2(8) = 10$.

In a recent article, Andrews, Fraenkel and Seller, see [**3**], provided the following beautiful characterization of the number of m-ary partitions mod m relying only on the base-m representation of a number.

2010 AMS *Mathematics subject classification.* Primary 05A17, 11P83.
Keywords and phrases. Partitions, m-ary partitions, congruence properties.
Received by the editors on January 20, 2016.
DOI:10.1216/RMJ-2017-47-6-1

Theorem 1.1 (**[3]**). *If $m \geq 2$ and $n = (n_0, n_1, \ldots, n_k)_m$, then*

$$b_m(mn) = \prod_{i=0}^{k}(n_i + 1) \bmod m.$$

Their elegant proof follows from clever manipulation of power series and the generating function for m-ary partitions. Their result allows for a uniform proof of many known congruence properties of m-ary partitions, originally conjectured by Churchhouse and proved by Rødseth, Andrews and Gupta, see [**1, 6, 8, 9, 10**].

Theorem 1.1 implies that

$$b_m(mn) - \prod_{i=0}^{k}(n_i + 1) = m \cdot q$$

for some $q \in \mathbb{N}$. Our primary result (Theorem 1.2) provides a combinatorial interpretation for the value of q. Furthermore, as a corollary to our main result, we obtain a new proof of Theorem 1.1 which does not rely on generating functions.

Note that the product in Theorem 1.1,

$$\prod_{i=0}^{k}(n_i + 1),$$

arises in various other places; for instance, when m is prime, this number counts the nonzero entries in row n of Pascal's triangle mod m, see [**7**]. This product may also be interpreted in terms of a partial order on the natural numbers arising from base-m representations. In particular, for fixed $m \geq 2$, we let $\ll_m$ represent the *m-dominance order* defined by $a \ll_m b$ if $a_i \leq b_i$ for all i, where $a = (a_0, a_1, \ldots, a_k)_m$ and $b = (b_0, b_1, \ldots, b_l)_m$, see [**4, 5**]. Then, for $n = (n_0, n_1, \ldots, n_k)_m$, the same product counts the number of integers dominated by n, see [**4**]. We will use the interpretation of the product in terms of the m-dominance order in what follows.

Now, let n be a positive integer with $m^k \leq n < m^{k+1}$. Then, every m-ary partition is of the form

$$\ell_k \cdot m^k + \ell_{k-1} \cdot m^{k-1} + \cdots \ell_1 \cdot m + \ell_0$$

with $\ell_i \geq 0$ for all i. We will denote such a partition by $[\ell_0, \ell_1, \ldots, \ell_{k-1}, \ell_k]_m$. It is noteworthy to mention here that the base-m representation of n yields an m-ary partition

$$(n_0, n_1, \ldots, n_k)_m \longmapsto [n_0, n_1, \ldots, n_k]_m.$$

Finally, we define a function *nops* from m-ary partitions of n to $\mathbb{N}$ by

$$nops([\ell_0, \ell_1, \ldots, \ell_{k-1}, \ell_k]_m) = \sum_{i=0}^{k} \ell_i;$$

this represents the number of parts of the partition.

Now, let $n = (n_0, n_1, \ldots, n_k)_m$. We call an m-ary partition, ℓ, of n *simple* if $\ell = [\ell_0, \ell_1, \ldots, \ell_k]_m$ with $\ell_i \leq n_i$ for all $i \geq 1$. Thus, simple partitions are obtained by replacing powers of m in the m-ary representation with the appropriate number of 1s. Let $P_m(n)$ be the set of m-ary partitions of n, $S_m(n)$ the set of simple m-ary partitions of n and $N_m(n) = P_m(n) \setminus S_m(n)$ the set of *non-simple* m-ary partitions of n. Restricting the function *nops* to $N_m(n)$, we obtain the following result.

Theorem 1.2. *Let $m \geq 2$ and $n \in \mathbb{N}$. Then, the nops function is equidistributed* modulo m *on the set $N_m(n)$.*

As a corollary, we obtain the following.

Corollary 1.3. *Let $b_m(n)$ be the number of m-ary partitions of $n = (n_0, n_1, \ldots, n_k)_m$. Then*

$$b_m(n) \equiv \prod_{i=1}^{k} (n_i + 1) \bmod m.$$

Note that the previous corollary is stated slightly differently than Theorem 1.1, which is given only for $b_m(mn)$; however, due to the fact that $b_m(mn+r) = b_m(mn)$ when $0 < r < m$ (as stated in **[3]**), the two forms are equivalent.

This paper is organized as follows. Section 2 contains the details necessary to prove Theorem 1.2. We prove the theorem and its corollary in Section 3. In addition, we use Theorem 1.2 to prove that the *nops* function is equidistributed mod m on the entire set of m-ary partitions, $P_m(n)$, if and only if $m-1$ appears in the base-m representation of n, see Theorem 3.2. Section 4 contains a detailed example illustrating the results in Sections 2 and 3. Finally, in Section 5, we describe some possible extensions.

2. Technical details. In this section, we provide a systematic method for partitioning $N_m(n)$, which will be used to prove Theorem 1.2. We have included a detailed example of this method of partitioning in Section 4.

Let $m \geq 2$ and $n \in \mathbb{N}$ be fixed with $n = (n_0, n_1, \ldots, n_k)_m$. First, we define a function $f_{m,n} : N_m(n) \to \mathbb{N}$ by

$$f_{m,n}([\ell_0, \ell_1, \ldots, \ell_k]_m) = (b_0, b_1, b_2, \ldots, b_k)_m,$$

where $b_i = \min(n_i, \ell_i)$ for all i; note that $b_0 = n_0$ since $\ell_0 \equiv n_0 \pmod{m}$. The next lemma follows by construction.

Lemma 2.1. *For any non-simple partition* $\ell \in N_m(n)$*, we have* $f_{m,n}(\ell) \ll_m n$*.*

Now, we use $f_{m,n}$ to define a relation on $N_m(n)$ by $\rho \sim \gamma$ if $f_{m,n}(\rho) = f_{m,n}(\gamma)$.

Lemma 2.2. *The relation* $\sim$ *is an equivalence relation, and thus,*

$$\{f_{m,n}^{-1}(b) \mid b \in \mathbb{N} \text{ and } b \ll_m n \text{ and } f_{m,n}^{-1}(b) \neq \emptyset\}$$

forms a partition of $N_m(n)$*.*

Proof. Any function yields such an equivalence relation. □

Lemma 2.3. *Let* ℓ *be a non-simple* m*-ary partition of* n*. Then* ℓ *can be component-wise decomposed as*

$$\ell = [\ell_0, \ell_1, \ldots, \ell_k]_m = [r_0, r_1, \ldots, r_k]_m + [b_0, b_1, b_2, \ldots, b_k]_m,$$

where $b = (b_0, b_1, b_2, \ldots, b_k)_m = f_{m,n}(\ell)$ *and* $r_i \geq 0$ *for all* i*. Moreover, it follows that* $r_i > 0$ *only if* $n_i = b_i$*.*

Proof. Since $r_i = \ell_i - \min(\ell_i, n_i)$, it is clear that $r_i \geq 0$. Now, if $r_i > 0$, then $\min(\ell_i, n_i) \neq \ell_i$ so that $b_i = n_i$, as required. □

Lemma 2.4. *Let ℓ be a non-simple m-ary partition of $n = (n_0, n_1, \ldots, n_k)_m$ with $\ell \in f_{m,n}^{-1}(b)$ where $b = (b_0, b_1, \ldots, b_k)_m$. Suppose that ℓ is of the form*

$$\ell = [\ell_0, b_1, b_2, \ldots, b_{j-1}, \ell_j, \ell_{j+1}, \ldots, \ell_k]_m$$

with $\ell_j > n_j = b_j$. Then, there is a unique pair (r, h) with $r \geq 1$ and $0 \leq h < m^j$ such that $\ell_j \leq n_j + mr$, there is an m-ary partition of the form $[h, b_1, b_2, \ldots, b_{j-1}, b_j + mr, \ell_{j+1}, \ldots, \ell_k]_m$, and there is no m-ary partition of the form $[h', b_1, b_2, \ldots, b_{j-1}, g, \ell_{j+1}, \ldots, \ell_k]_m$ with $g > b_j + mr$.

Proof. Let $s = \ell_j - b_j = \ell_j - n_j > 0$. According to the division algorithm, there is a unique h satisfying $\ell_0 = t \cdot m^j + h$ where $0 \leq h < m^j$. Then, clearly,

$$[h, b_1, b_2, \ldots, b_{j-1}, b_j + s + t, \ell_{j+1}, \ldots, \ell_k]_m$$

is an m-ary partition of n. Note that

$$[h', 0, 0, \ldots, 0, b_j + s + t, \ell_{j+1}, \ldots, \ell_k]_m$$

is an m-ary partition of n where

$$h' := h + \sum_{i=1}^{j-1} b_i = \sum_{i=0}^{j-1} n_i < m^j.$$

This implies that

$$[0, 0, 0, \ldots, 0, b_j + s + t, \ell_{j+1}, \ldots, \ell_k]_m$$

is an m-ary partition of $n' = (0, 0, \ldots, 0, n_j, n_{j+1}, \ldots, n_k)_m$. However, since $n_j = b_j$ and $s + t > 0$, then

$$0 < s + t = \sum_{i=j+1}^{k} (n_i - \ell_i) \cdot m^{i-j}.$$

Thus, $s + t = mr$ for some $r \geq 1$, as required. Finally, we see that $b_j + mr$ is the largest number of parts of the form m^j we can have without reducing some ℓ_i with $i > j$. □

Corollary 2.5. *Let ℓ be a non-simple m-ary partition of $n = (n_0, n_1, \ldots, n_k)_m$ with $\ell \in f_{m,n}^{-1}(b)$ where $b = (b_0, b_1, \ldots, b_k)_m$. Suppose that ℓ is of the form*

$$\ell = [\ell_0, b_1, b_2, \ldots, b_{j-1}, \ell_j, \ell_{j+1}, \ldots, \ell_k]_m$$

with $\ell_j > n_j = b_j$. Then, there is an m-ary partition of the form $[v, b_1, b_2, \ldots, b_{j-1}, u, \ell_{j+1}, \ldots, \ell_k]_m$ for all $b_j < u \leq b_j + mr$ where r is given by Lemma 2.4.

Proof. Let $b_j < u \leq b_j + mr$, and consider the partition of the form $\rho = [h, b_1, b_2, \ldots, b_{j-1}, b_j + mr, \ell_{j+1}, \ldots, \ell_k]_m$ guaranteed by Lemma 2.4. Then we find y such that $b_j + mr = u + y$ where $y \geq 0$. Next, construct an m-ary partition from ρ by converting y parts of the form m^j to $y \cdot m^j$ parts of the form m^0, obtaining the partition

$$[h + y \cdot m^j, b_1, b_2, \ldots, b_{j-1}, u, \ell_{j+1}, \ldots, \ell_k]_m,$$

as required. □

Now, fix $b \ll_m n$ with $f_{m,n}^{-1}(b) \neq \emptyset$. For each $1 \leq z \leq k$, we define

$$B(z) := \{\rho \in f_{m,n}^{-1}(b) \mid \min\{i \geq 1 \mid \rho_i \neq b_i\} = z\}.$$

Again, the following lemma is clear by construction.

Lemma 2.6. *Let $b \ll_m n$ with $f_{m,n}^{-1}(b) \neq \emptyset$. Then, the collection of sets $\{B(z) \mid B(z) \neq \emptyset\}$ forms a partition of $f_{m,n}^{-1}(b)$.*

As our final step, we fix z with $1 \leq z \leq k$ such that $B(z) \neq \emptyset$. Now, we define a relation on $B(z)$ as follows. We say that $\rho \simeq_{b,z} \gamma$ if $\gamma_i = \rho_i$ for all $i > z$.

Lemma 2.7. *The relation $\simeq_{b,z}$ on $B(z)$ is an equivalence relation and thus provides a partition of $B(z)$.*

Proof. This is again clear by construction. □

Proposition 2.8. *Let $n \in \mathbb{N}$, $b \in \mathbb{N}$ with $b \ll_m n$ and $1 \leq z \leq k$ be such that $f_{m,n}^{-1}(b) \neq \emptyset$ and $B(z) \neq \emptyset$. Then, the nops function is equidistributed* modulo *m on each equivalence class of $\simeq_{b,z}$.*

Proof. Suppose that C is an equivalence class of $\simeq_{b,z}$. Then, by construction, there exists an $\ell_{z+1}, \ell_{z+2}, \ldots, \ell_k$ such that every partition in C is of the form

$$[h, b_1, b_2, \ldots, b_{z-1}, h', \ell_{z+1}, \ell_{z+2}, \ldots, \ell_k]_m$$

for some h and h' with $h' > b_z$. Now, according to Lemma 2.5 and Corollary 2.5, there exists some $r \geq 1$ such that

$$C = \{[h, b_1, b_2, \ldots, b_{z-1}, u, \ell_{z+1}, \ell_{z+2}, \ldots, \ell_k]_m \mid h \in \mathbb{N} \text{ and } b_j < u \leq b_j + mr\}.$$

Thus, $|C| = mr$. Now, for each $1 \leq w \leq m$, we define

$$C_w = \{[h_j, b_1, b_2, \ldots, b_j + w + jm, \ell_{z+1}, \ell_{z+2}, \ldots, \ell_k]_m \mid 1 \leq j \leq (r-1)\},$$

and we note that $|C_w| = r - 1$ for all w and the set $\{C_w\}$ forms a partition of C. Moreover, for each w, $nops(\gamma) \equiv nops(\rho) \pmod m$ for all $\gamma, \rho \in C_w$, and $nops(\rho) \equiv nops(\gamma) + 1 \pmod m$ whenever $\gamma \in C_w$ and $\rho \in C_{w+1}$. $\square$

3. Proof of Theorem 1.2 and consequences.

Proof of Theorem 1.2. Let $b \ll_n n$ with $f_{m,n}^{-1}(b) \neq \emptyset$. Then, let $1 \leq z \leq k$ with $B(z)$ be non-empty. By Proposition 2.8 and Lemma 2.7, the *nops* function is equidistributed mod m on $B(z)$. Likewise, by Lemma 2.6, the *nops* function is equidistributed mod m on $f_{m,n}^{-1}(b)$. Finally, Lemma 2.2 implies that the *nops* function is equidistributed mod m on $N_m(n)$. $\square$

Let $n = (n_0, n_1, \ldots, n_k)_m$. Then, according to Theorem 1.2, $N_m(n) = m \cdot q$, where q is the number of non-simple m-ary partitions with the number of parts divisible by m. However, it is clear that there is a bijection between simple m-ary partitions of n and the integers equivalent to n mod m that are m-dominated by n:

$$[\ell_0, b_1, b_2, \ldots, b_k]_m \longleftrightarrow (n_0, b_1, b_2, \ldots, b_k)_m.$$

As previously mentioned, there are $\prod_{i=1}^{k}(n_i + 1)$ integers equivalent to n mod m that are m-dominated by n (see **[4]** and use the fact that b is equivalent to $n \bmod m$ if and only if $b_0 = n_0$). Thus, we see that

$$b_m(n) = |N_m(n)| + |S_m(n)| = m \cdot q + \prod_{i=1}^{k}(n_i + 1);$$

therefore, Corollary 1.3 holds.

Understanding the *nops* function on $N_m(n)$ allows us to characterize when the *nops* function is equidistributed $\bmod\, m$ on the entire set of m-ary partitions, $P_m(n)$.

Corollary 3.1. *The nops function is equidistributed* modulo m *on* $P_m(n)$ *if and only if nops is equidistributed* modulo m *on the simple* m*-ary partitions,* $S_m(n)$*.*

Proof. This follows from Theorem 1.2 since $P_m(n)$ is the disjoint union of $N_m(n)$ and $S_m(n)$. □

Theorem 3.2. *Let* $m \geq 2$*, and let* $n = (n_0, n_1, \ldots, n_k)_m$ *be the base-*m *representation of* n*. Then, the nops function is equidistributed* modulo m *on* $P_m(n)$ *if and only if the set* $\{n_1, n_2, \ldots, n_k\}$ *contains* $m - 1$*.*

Proof. First, suppose that $n_i = m - 1$ for some $i \geq 1$. Due to Corollary 3.1, we need to show that the *nops* function is equidistributed on $S_m(n)$. Now, for each $w \in \{0, 1, \ldots, m-1\}$, let

$$A_w = \{\ell \in S_m(n) \mid \ell_i = w\}.$$

Then, it is clear that $\{A_w \mid w \in \{0, 1, \ldots, m-1\}\}$ forms a set partition of $S_m(n)$. Furthermore, since all the m-ary partitions in A_w are simple, there is a bijection $g_{w,w'} : A_w \to A_{w'}$ given by

$$g_{w,w'}((\ell_0, \ell_1, \ldots, w, \ldots, \ell_k)) := (\ell_0 + (w - w') \cdot m^i, \ell_1, \ldots, w', \ldots, \ell_k)$$

such that $|A_w| = |A_{w'}|$ for all $w, w' \in \{0, 1, \ldots, m-1\}$. Finally, let $\ell \in A_0$. Then, for each $w \in \{0, 1, \ldots, m-1\}$, we have $nops(g_{0,w}(\ell)) \equiv nops(\ell) + w \,(\bmod\, m)$. Thus, the *nops* function is equidistributed $\bmod m$ on $S_m(n)$.

Conversely, suppose that $m - 1 \notin \{n_1, \ldots, n_k\}$. First, assume that the only nonzero base-m digits are n_0 and n_k so that, by assumption,

$n_k \leq m-2$. Then, there are only $n_k+1 \leq m-1$ simple partitions, and thus, the *nops* function cannot be equidistributed mod m on $S_m(n)$. Next, assume that $0 < n_j \leq m-2$ for some $1 \leq j < k$. Similar to the previous paragraph, for each $w \in \{0, 1, \ldots, n_j\}$, let

$$A_w = \{\ell \in S_m(n) \mid \ell_j = w\}.$$

As before, $|A_w| = |A_{w'}|$ for all $w, w' \in \{0, 1, \ldots, n_j\}$ and, for each $\ell \in A_0$ and each $w \in \{0, 1, \ldots, n_j\}$, we have $nops(g_{0,w}(\ell)) \equiv nops(\ell) + w \pmod{m}$. Since $n_j \leq m-2$, the *nops* function will be equidistributed mod m on $S_m(n)$ if and only if the *nops* function is equidistributed mod m on A_0. However, we see that there is a bijection $h : A_0 \to S_m(n - n_j \cdot m^j)$ given by

$$h((\ell_0, \ell_1, \ldots, 0, \ldots, \ell_k)) := (\ell_0 - n_j \cdot m^j, \ell_1, \ldots, 0, \ldots, \ell_k).$$

Moreover, we note that $nops(h(\ell)) \equiv nops(\ell) \pmod{m}$ such that *nops* is equidistributed mod m on A_0 if and only if *nops* is equidistributed mod m on $S_m(n - n_j \cdot m^j)$, which implies that *nops* is equidistributed mod m on $S_m(n)$ if and only if *nops* is equidistributed mod m on $S_m(n - n_j \cdot m^j)$. Since the digit sets of n and $n - n_j \cdot m^j$ are identical except in position j, we can use this argument to deduce that *nops* is equidistributed mod m on $S_m(n)$ if and only if *nops* is equidistributed mod m on $S_m(n - \sum_{i=1}^{k-1} n_i \cdot m^i)$. However,

$$n - \sum_{i=1}^{k-1} n_i \cdot m^i = (n_0, 0, \ldots, 0, n_k)$$

and $n_k \leq m-2$; in this case, we have already shown that *nops* is not equidistributed mod m on $S_m(n - \sum_{i=1}^{k-1} n_i \cdot m^i)$. The result follows. □

4. Detailed example. We illustrate the results of the previous two sections with an example. Let $m = 3$, and consider $n = 60 = (0, 2, 0, 2)_3$. Then, the total number of 3-ary partitions of 60 is 117, i.e., $b_3(60) = 117$. Of these 117, there are 9 simple partitions listed in Figure 1.

In Figures 2–7, we list the remaining 108 non-simple partitions, those in $N_3(60)$, using the results in Section 2. The numbers 3-dominated by 60 are

$$0,\ 3,\ 6,\ 27,\ 30,\ 33,\ 54,\ 57,\ 60.$$

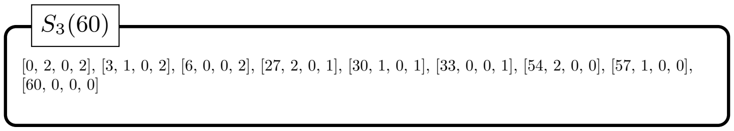

FIGURE 1.

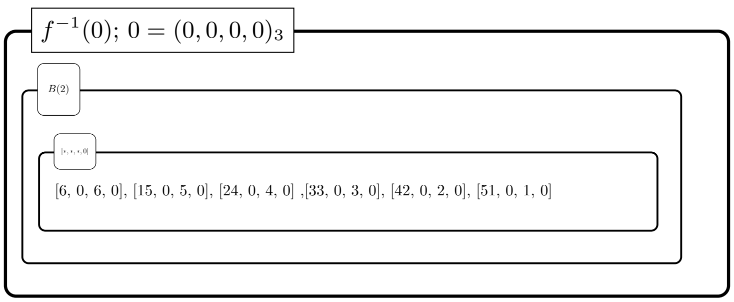

FIGURE 2.

Let f represent $f_{3,60}$. It turns out that $f^{-1}(54)$, $f^{-1}(57)$ and $f^{-1}(60)$ are all empty. There are 6 partitions in $f^{-1}(0)$ and $f^{-1}(3)$; there are 69 partitions in $f^{-1}(6)$; there are 3 partitions in $f^{-1}(27)$ and $f^{-1}(30)$; and there are 21 partitions in $f^{-1}(33)$. All of the nonempty inverse images are listed in Figures 2–7. The subsets correspond to the nonempty sets $B(z)$ for $1 \leq z \leq 3$, and then, the subsets of $B(z)$ correspond to the partition given by $\simeq_{b,z}$ guaranteed by Lemma 2.7. The most representative example is that of $f^{-1}(6)$ as it contains both $B(1)$ and $B(2)$ ($B(3) = \emptyset$) and $B(1)$ is further partitioned into six equivalence classes for $\simeq_{6,1}$.

We can then check that the cardinality of each of the equivalence classes of $\simeq_{b,z}$ is a multiple of 3, and the *nops* function is equidistributed $\bmod 3$ on these smallest parts (see the proof of Theorem 1.2), thus showing that the *nops* function is equidistributed on $N_3(60)$.

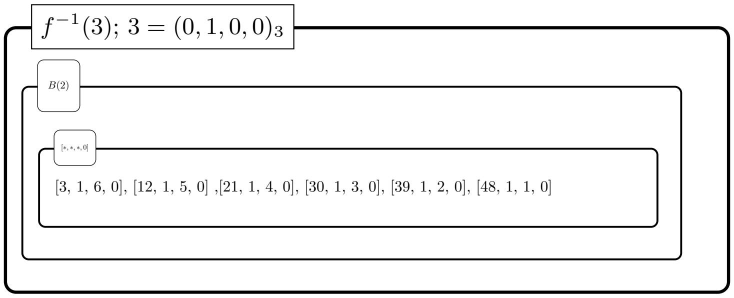

FIGURE 3.

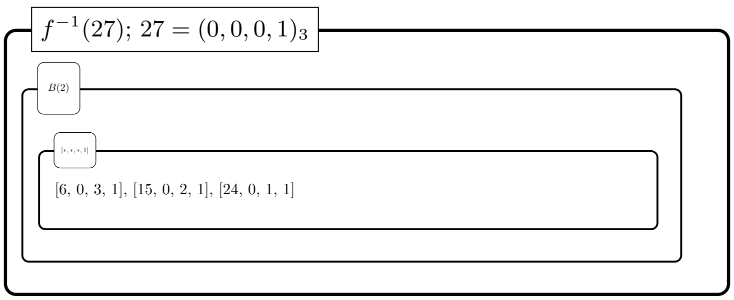

FIGURE 4.

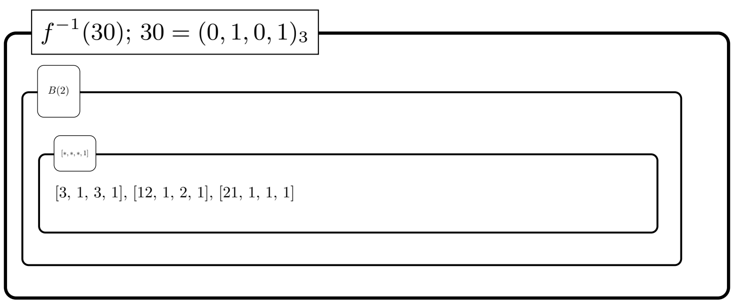

FIGURE 5.

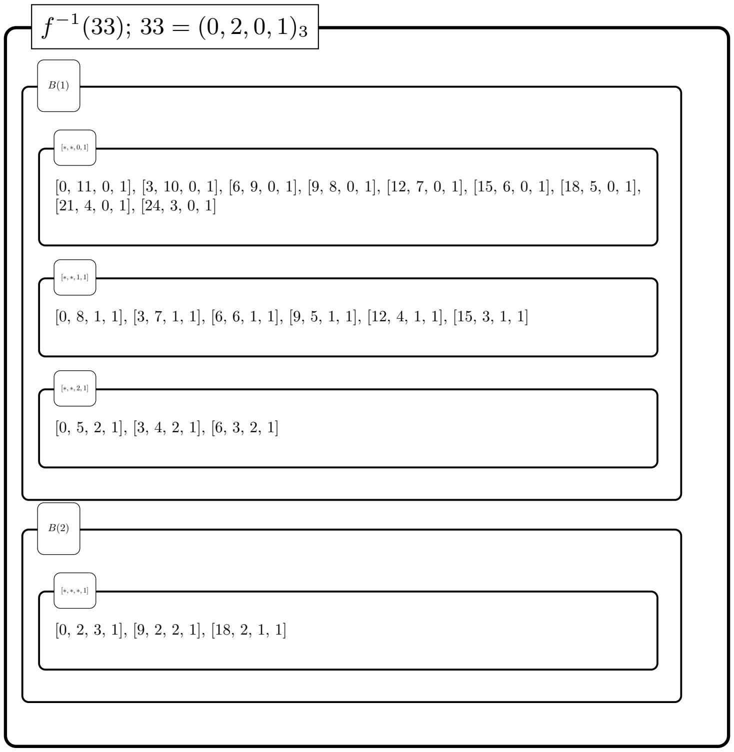

FIGURE 6.

5. Extensions. In this section, we briefly discuss a possible way of extending these results to other congruence relations. We note that the set of non-simple m-ary partitions $N_m(n)$ can be defined as

$$N_m(n) = \{\ell \in P_m(n) \mid \ell_j > n_j \text{ for some } j \geq 1\},$$

where $n = (n_0, \ldots, n_k)_m$ is the base-m representation of n. Consider the following generalizations. For any $c \geq 1$, we let

$$N_{m,c} = \{\ell \in P_m(n) \mid \ell_j > n_j,\ \ell_{j+1} = n_{j+1}, \ldots, \ell_{j+c} = n_{j+c} \text{ for some } j \geq 1\},$$

$f^{-1}(6)$; $6 = (0, 2, 0, 0)_3$

$B(1)$

$[*, *, 0, 0]$

[0, 20, 0, 0], [3, 19, 0, 0], [6, 18, 0, 0], [9, 17, 0, 0], [12, 16, 0, 0],
[15, 15, 0, 0], [18, 14, 0, 0], [21, 13, 0, 0], [24, 12, 0, 0], [27, 11, 0, 0], [30, 10, 0, 0],
[33, 9, 0, 0], [36, 8, 0, 0], [39, 7, 0, 0], [42, 6, 0, 0],
[45, 5, 0, 0], [48, 4, 0, 0], [51, 3, 0, 0]

$[*, *, 1, 0]$

[0, 17, 1, 0], [3, 16, 1, 0], [6, 15, 1, 0], [9, 14, 1, 0], [12, 13, 1, 0],
[15, 12, 1, 0], [18, 11, 1, 0], [21, 10, 1, 0], [24, 9, 1, 0], [27, 8, 1, 0],
[30, 7, 1, 0], [33, 6, 1, 0], [36, 5, 1, 0], [39, 4, 1, 0], [42, 3, 1, 0]

$[*, *, 2, 0]$

[0, 14, 2, 0], [3, 13, 2, 0], [6, 12, 2, 0], [9, 11, 2, 0], [12, 10, 2, 0],
[15, 9, 2, 0], [18, 8, 2, 0], [21, 7, 2, 0], [24, 6, 2, 0], [27, 5, 2, 0],
[30, 4, 2, 0], [33, 3, 2, 0]

$[*, *, 3, 0]$

[0, 11, 3, 0], [3, 10, 3, 0], [6, 9, 3, 0], [9, 8, 3, 0], [12, 7, 3, 0],
[15, 6, 3, 0], [18, 5, 3, 0], [21, 4, 3, 0], [24, 3, 3, 0]

$[*, *, 4, 0]$

[0, 8, 4, 0], [3, 7, 4, 0], [6, 6, 4, 0], [9, 5, 4, 0], [12, 4, 4, 0], [15, 3, 4, 0]

$[*, *, 5, 0]$

[0, 5, 5, 0], [3, 4, 5, 0], [6, 3, 5, 0]

$B(2)$

$[*, *, *, 0]$

$[0, 2, 6, 0]$, $[9, 2, 5, 0]$, $[18, 2, 4, 0]$, $[27, 2, 3, 0]$, $[36, 2, 2, 0]$, $[45, 2, 1, 0]$

Figure 7.

where we note that $N_m(n)$ can be interpreted as $N_{m,0}$. Then, we prove an analogous result to Lemma 2.4 that shows $|N_{m,c}| \equiv 0 \pmod{m^{c+1}}$. Therefore, if we can determine the size of the set

$$S_{m,c}(n) := P_m(n) \setminus N_{m,c}(n)$$

using mere knowledge of n (possibly the base-m representation of n), then we obtain interesting congruence properties for $b_m(n)$ mod m^{c+1} for any c.

REFERENCES

1. George E. Andrews, *Congruence properties of the m-ary partition function*, J. Number Theory **3** (1971), 104–110.

2. ______, *The theory of partitions*, Cambr. Math. Libr., Cambridge University Press, Cambridge, 1998.

3. George E. Andrews, Aviezri S. Fraenkel and James A. Sellers, *Characterizing the number of m-ary partitions modulo m*, Amer. Math. Month. **122** (2015), 880–885.

4. Tyler Ball, Tom Edgar and Daniel Juda, *Dominance orders, generalized binomial coefficients, and Kummer's theorem*, Math. Mag. **87** (2014), 135–143.

5. Tyler Ball and Daniel Juda, *Dominance over* ℕ, Rose Hulman Undergrad. Math. J. **14** (2013).

6. R.F. Churchhouse, *Congruence properties of the binary partition function*, Proc. Cambr. Philos. Soc. **66** (1969), 371–376.

7. N.J. Fine, *Binomial coefficients modulo a prime*, Amer. Math. Month. **54** (1947), 589–592.

8. Hansraj Gupta, *On m-ary partitions*, Proc. Cambr. Philos. Soc. **71** (1972), 343–345.

9. ______, *A simple proof of the Churchhouse conjecture concerning binary partitions*, Indian J. Pure Appl. Math. **3** (1972), 791–794.

10. Øystein Rødseth, *Some arithmetical properties of m-ary partitions*, Proc. Cambr. Philos. Soc. **68** (1970), 447–453.

Pacific Lutheran University, Department of Mathematics, Tacoma, WA 98447
Email address: **edgartj@plu.edu**