A CASE-FREE CHARACTERIZATION OF HYPERBOLIC COXETER SYSTEMS

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Abstract. We provide a case-free characterization of hyperbolic Coxeter systems depending only on the Coxeter graph. Consequently, we uncover a case-free proof of the fact that every infinite, non-affine Coxeter system contains a standard parabolic subsystem that is a hyperbolic Coxeter system.

1. Introduction and Preliminaries

The standard classification of finite reflection groups (or finite Coxeter systems) and affine Coxeter systems can be found in [5] or [6]. Affine Coxeter systems are in some natural sense the smallest infinite Coxeter groups (e.g. they have only polynomial growth). In this note, we will consider the "next" class of infinite Coxeter systems known as hyperbolic Coxeter systems. There is a very well known, case-by-case, classification of hyperbolic Coxeter systems, which can be found in [5, chapter 6]. We discuss several characterizations of hyperbolic Coxeter systems by properties of their Coxeter graphs. According to [2], it is well known that any irreducible, infinite, non-affine Coxeter system contains a standard parabolic subsystem that is a hyperbolic Coxeter system; however, we could not find a proof in the literature, and the proof of the characterizations we include allows us to provide a case-free proof of this fact.

Remark. The author needed a proof of this fact to prove some results about large reflection subsystems and to examine the growth types of Coxeter systems in [3]. However, we believe that this simple proof may be of broader interest. During the preparation of this note, [1] was posted on the arXiv and has similar results proved by different methods.

1.1. Let (W,S) be a finite rank Coxeter system where $m(s_i,s_j)$ represents the order of s_is_j for all i,j. Without loss of generality, we will assume that (W,S) is realized in its standard reflection representation

Date: November 9, 2010.

on a real vector space V with Π denoting the set of simple roots and $(-,-) : V \times V \to \mathbb{R}$ representing the associated bilinear form given by $(\alpha,\beta) = -\cos\frac{\pi}{m(s_\alpha,s_\beta)}$ if $m(s_\alpha,s_\beta) \neq \infty$ and $(\alpha,\beta) \leq -1$ otherwise. Then we denote the roots and positive roots by Φ and Φ^+ respectively. Let $A := A_{(W,S)}$ be the matrix of the bilinear form $(-,-)$. That is, A is a symmetric matrix with $A_{i,j} := A_{s_i,s_j} = (\alpha_i,\alpha_j)$ where α_k is the simple root associated to s_k.

We let $\Gamma := \Gamma_{(W,S)}$ be the associated Coxeter graph; Γ is the undirected, labeled graph with vertex set S and edges (s_i,s_j) whenever $m(s_i,s_j) \geq 3$ and each edge labeled by the corresponding order $m(s_i,s_j)$. We say that (W,S) is irreducible if Γ is connected. For $J \subseteq S$, let Γ_J be the full subgraph of Γ with vertices corresponding to J. We may abuse notation and say that $s \in S - J$ is connected to J if s is connected to Γ_J.

1.2. For any real symmetric matrix over a finite dimensional vector space, we define the inertia of A to be the triple (a,b,c) where a represents the number of positive eigenvalues of A, b represents the number of negative eigenvalues of A, and c is the number of zero eigenvalues of A. We will abuse notation and say that a Coxeter system has inertia (a,b,c) if the corresponding matrix A has inertia (a,b,c). We say that an $n \times n$ symmetric matrix A is positive definite if it has inertia $(n,0,0)$. We say that a $n \times n$ symmetric matrix is positive semidefinite if it has inertia $(a,0,c)$, i.e. it has only non-negative eigenvalues. If A is positive definite or positive semi-definite then we say A is of positive type. We state the following fact of real symmetric matrices (see for example [4]):

1.3. Proposition (Sylvester's law of inertia)**.** *If A is an $n \times n$ real symmetric matrix with inertia (a,b,c), then A is congruent to the diagonal matrix $D = I_a \oplus -I_b \oplus \mathbf{0}_c$.*

1.4. Let (W,S) be an irreducible Coxeter system with associated graph Γ and matrix A. Following [5], we know that W is finite if and only if A is positive definite. Also, we know that W is affine if and only if the inertia is $(n-1,0,1)$. We define an irreducible Coxeter system to be *hyperbolic* if it has inertia $(n-1,1,0)$ and $(v,v) < 0$ for all $v \in C := \{\lambda \in V | \ (\lambda,\alpha_s) > 0 \ \forall s \in S\}$. Also, we have the following characterization of hyperbolic Coxeter systems.

1.5. Proposition. [5, Prop. 6.8] *Let (W,S) be an irreducible Coxeter system, with graph Γ and associated bilinear form $(-,-)$ with matrix A. Then (W,S) is hyperbolic if and only if the following two conditions are satisfied:*

(1) $(-,-)$ *(or equivalently A) is non-degenerate, but not positive definite.*
(2) *For each* $s \in S$*, the Coxeter graph* Γ'*, obtained by removing* s *from* Γ*, is of positive type.*

2. Statement of the Main Results

Proposition 1.5 is used to provide list of all hyperbolic Coxeter systems. We intend to prove the following characterization of hyperbolic Coxeter systems that only depends on the structure of the Coxeter graph and does not require case-by-case examination.

2.1. Theorem. *Let* (W, S) *be an irreducible, finite rank, infinite, non-affine Coxeter system with Coxeter graph* Γ*. Then the following hold.*

(1) *If every proper subgraph* $\Gamma' \subset \Gamma$ *is the disjoint union of graphs that are Coxeter graphs of only finite or affine type, then* (W, S) *is hyperbolic.*
(2) *Suppose* (W, S) *satisfies one of the following conditions.*
   (a) *There exists* $s \in S$ *such that* $\Gamma - \{s\}$ *is a disjoint union of Coxeter graphs of finite type.*
   (b) $\Gamma - \{s\}$ *is a connected Coxeter graph of affine type for all* $s \in S$*.*

   Then (W, S) *is hyperbolic.*

The proof of this theorem will also provide us with a proof of the following (known) fact.

2.2. Theorem. *Any finite rank, irreducible, infinte, non-affine Coxeter system contains a standard parabolic subsystem of hyperbolic type.*

3. Proof of the Main Results

3.1. From this point on, we may use (W, S), W and $\Gamma = \Gamma_{(W,S)}$ interchangeably to represent the corresponding Coxeter system. We say a graph Γ is of finite (resp. affine) type if the corresponding Coxeter system is finite (resp. affine).

We begin by proving a technical lemma that will be needed in what is to follow.

3.2. Lemma. *Suppose that* (W, S) *is an infinite, irreducible, non-affine Coxeter system. Let* $|S| = n$*. Suppose that there exists* $s \in S$ *such that the standard parabolic subsystem* $(W_{S\setminus\{s\}}, S \setminus \{s\})$ *is a connected affine Coxeter system. Then the inertia of* (W, S) *is* $(n-1, 1, 0)$*.*

Proof. Let (W, S) be as in the statement. Thus, there exists $s \in S$ such that $S \setminus \{s\}$ is of connected affine type. By removing another simple reflection s' we must get a finite Coxeter system of rank $n-2$, and so (W, S) must have at least $n-2$ positive eigenvalues.

Now, we order S so that $S = \{s_1, ..., s_{n-1}, s_n\}$ where $s_n = s$. Then, we let A' be the $(n-1)\times(n-1)$ leading principal minor of the matrix A associated to (W, S) and Γ. Since A' is the matrix for $(W_{S\setminus\{s\}}, S \setminus \{s\})$ which is of affine type of rank $n-1$, it has inertia $(n-2, 0, 1)$ and there exists an isotropic root called δ. Additionally, we know that $\delta = \sum_{i=1}^{n-1} k_i \alpha_{s_i}$ with $k_i > 0$ for all i (c.f. [6]). By Proposition 1.3, we know that there exists an invertible matrix C such that $C^{tr}A'C = I_{n-2} \oplus \mathbf{0}_1$. We can lift this to a matrix $D = C \oplus I_1$ so that

$$B := D^{tr}AD = \begin{pmatrix} \begin{matrix} & & \\ & I_{n-2} & \\ & & \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \begin{matrix} a_1 \\ \vdots \\ a_{n-2} \end{matrix} \\ 0 \cdots 0 & 0 & c \\ a_1 \cdots a_{n-2} & c & 1 \end{pmatrix}$$

where $c = (\alpha_s, \delta)$ and $a_i \in \mathbb{R}$ for all i. Since (W, S) is irreducible, Γ is connected and so there exists some $i \neq n$ such that $(\alpha_{s_n}, \alpha_{s_i}) < 0$. Thus by the characterization of δ above we see that $c = (\alpha_{s_n}, \delta) \neq 0$.

Next, we want to find the determinant of B. To do this, we expand the matrix along the $(n-1)$'st row to get that $\det(B) = -c \cdot \det(B')$ where

$$B' := \begin{pmatrix} \begin{matrix} & & \\ & I_{n-2} & \\ & & \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ a_1 \cdots a_{n-2} & c \end{pmatrix}$$

Now, we can find the determinant of B' easily by expanding along the $(n-1)$'st column of B'. This gives us that $\det(B') = c\det(I_{n-2})$. Putting the results together we see that

$$\det(B) = -c\det(B') = -c^2\det(I_{n-2}) = -c^2.$$

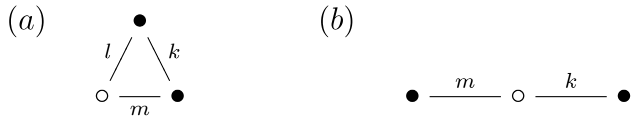
Since B has at least $n-2$ positive eigenvalues and negative determinant (because $c \neq 0$), we know that exactly one of the remaining eigenvalues must be negative. Indeed, if both were negative or both were positive, the determinant of B would be positive. Thus B has inertia $(n-1, 1, 0)$. Following Proposition 1.3, A must have inertia $(n-1, 1, 0)$ as well. $\square$

3.3. We now define the two classes of Coxeter systems arising in Theorem 2.1. First, define $\mathbf{H}$ to be the class of all irreducible, finite rank, infinite, non-affine Coxeter systems, (W, S), with Coxeter graph Γ such that every proper subgraph $\Gamma' \subset \Gamma$ is the disjoint union of graphs that are Coxeter graphs of only finite or affine type. Next, define $\mathbf{h} \subseteq \mathbf{H}$ as the class of all Coxeter systems, (W, S), in $\mathbf{H}$ with Γ satisfying exactly one of the following:

(**h1**) There exists $s \in S$ such that $\Gamma - \{s\}$ is a disjoint union of Coxeter graphs of finite type.

(**h2**) $\Gamma - \{s\}$ is a connected Coxeter graph of affine type for all $s \in S$.

3.4. Example. Notice that any infinite, rank 3 Coxeter system can be described by one of the two following graphs (note: we color a vertex $\circ$ if we intend to refer to it below):



with $l \geq m \geq k$. For arbitrary m and k, if we remove vertex $\circ$ from graph (b) we will be left with the Coxeter graph for a dihedral Coxeter system of order 4. Thus, graph (b) is clearly of type (**h1**) (provided it is infinite and non-affine).

For graph (a), if $k \neq \infty$, then removing vertex $\circ$ leaves us with the Coxeter graph for a finite dihedral Coxeter system, and thus (a) is of type (**h1**) (provided it is not affine). If $k = \infty$ then we necessarily have $m = \infty$ and $l = \infty$. Therefore, removing any vertex from (a) gives us the graph $\bullet \overset{\infty}{\text{——}} \bullet$, which is clearly of affine type. Thus any infinite, non-affine irreducible rank 3 Coxeter system is in $\mathbf{h} \subseteq \mathbf{H}$.

The following lemmas allow us to prove the main result.

3.5. Lemma. *Every hyperbolic Coxeter system is in* $\mathbf{H}$.

Proof. By Proposition 1.5 part (2) we see that any hyperbolic Coxeter system must be in $\mathbf{H}$. In particular, the hyperbolic Coxeter systems are precisely those in $\mathbf{H}$ with inertia $(n-1, 1, 0)$. □

3.6. Lemma. *Any Coxeter system in* $\mathbf{h}$ *is hyperbolic.*

Proof. Let (W, S) be in $\mathbf{h} \subseteq \mathbf{H}$ and let $|S| = n$. Let Γ be the Coxeter graph of (W, S) and let A be the matrix associated to $(-,-)$.

Case 1: (W, S) is of type (**h1**). Then, since there exists $s \in S$ such that $(W_{S-\{s\}}, S - \{s\})$ is of finite type and so the associated bilinear form and matrix A' must have inertia $(n-1, 0, 0)$. Thus A must have at least $n-1$ positive eigenvalues. If A has inertia $(n, 0, 0)$ (resp. $(n-1, 0, 1)$) then (W, S) would be of finite type (resp. affine type) and thus $(W, S) \notin \mathbf{H}$, which is a contradiction (since $\mathbf{h} \subseteq \mathbf{H}$ by definition). Therefore A must have inertia $(n-1, 1, 0)$ forcing (W, S) to be hyperbolic by Proposition 1.5.

Case 2: (W, S) is of type (**h2**). Then since $S - \{s\}$ is of connected affine type Lemma 3.2 implies that the inertia of (W, S) is $(n-1, 1, 0)$. In particular $(-, -)$ is non-degenerate. Furthermore, since for each $r \in S$ we have $\Gamma - \{r\}$ is of positive type, Proposition 1.5 implies that (W, S) is hyperbolic.

□

3.7. Lemma. *Any infinite, irreducible, finite rank, non-affine Coxeter system* (W, S) *contains a standard parabolic subsystem in the class* **H**.

Proof. Let (W', S') be a minimal rank, irreducible, infinite, non-affine parabolic subsystem of (W, S), which must exist. Then since (W', S') is minimal rank, if we remove any vertex s then $(W_{S' \setminus \{s\}}, S' \setminus \{s\})$ cannot contain an infinite, non-affine parabolic subsystem. Thus, $(W_{S' \setminus \{s\}}, S' \setminus \{s\})$ must be of finite or affine type. □

3.8. Lemma. *Let* (W, S) *be in the class* **H**. *Then* (W, S) *is in the class* **h**.

Proof. Suppose that (W, S) is not of type (**h1**) and let $\Gamma := \Gamma_{(W,S)}$. Then because (W, S) is in class **H**, we know that $(W_{S \setminus \{s\}}, S \setminus \{s\})$ is of affine type for every $s \in S$. Suppose that $s \in S$ and $\Gamma_{S \setminus \{s\}}$ is not connected. Let $\Gamma_0, \Gamma_1, ..., \Gamma_n$ be the connected components (so $n \geq 1$). Now, since $\Gamma_{S \setminus \{s\}}$ is of affine type we can assume without loss of generality that Γ_0 is of affine type. Then $\Gamma' = \Gamma_0 \cup \{s\}$ is irreducible (since (W, S) is irreducible) and is properly contained in $\Gamma_0 \cup \{s\} \cup \{s_1\}$ where $s_1 \in \Gamma_1$. Thus Γ' is properly contained in Γ. However, Γ' cannot be of finite or affine type or else its subgraph Γ_0 would be finite, contradicting that Γ_0 is affine. Therefore if $\Gamma_{S \setminus \{s\}}$ is not connected then Γ properly contains Γ' which is infinite and non-affine, contradicting the assumption that (W, S) is in **H**. Thus, we have shown that if (W, S) is not of type (**h1**), then for every $s \in S$, $\Gamma \setminus \{s\}$ is of connected affine type, i.e. (W, S) is of type (**h2**). □

Finally, Theorem 2.1 and Theorem 2.2 clearly hold as follows.

Proof of Theorem 2.1. By Lemma 3.5, we know that all hyperbolic Coxeter systems are contained in **H**. By Lemma 3.8 we know that $\mathbf{H} \subseteq \mathbf{h}$ (and thus $\mathbf{h} = \mathbf{H}$). Finally, by Lemma 3.6 we know that every element of **h** is a hyperbolic Coxeter system. Thus we have equality throughout and the result follows. □

Proof of Theorem 2.2. This result follows by applying Theorem 2.1 along with Lemma 3.7. □

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