A classic problem in recreational mathematics involves adding the squares of the digits of a given number to obtain a new number. For example, from the number 49, we obtain $4^2 + 9^2 = 97$.

Although this action may seem a bit arbitrary, something interesting happens when we do it repeatedly. The sum of the squares of the digits in 97 is $9^2 + 7^2 = 130$. Continuing in this way, we obtain the sequence,

$$49 ightarrow 97 ightarrow 130 ightarrow 10 ightarrow 1 ightarrow 1 ightarrow \cdots$$

At this point, the sequence produces 1s forever.

The optimistic mathematician might hope that this will always happen, but alas, it doesn’t. For example, if we begin with 2, we obtain

$$2 ightarrow 4 ightarrow 16 ightarrow 37 ightarrow 58 ightarrow 89 ightarrow 145 ightarrow 42 ightarrow 20 ightarrow 4 \rightarrow \cdots$$

Once we reach 4, we enter a cycle that returns to 4 every eight steps.

Amazingly, these are the only possibilities: The sequence starting with any number leads to 1 or to the cycle

$$4 ightarrow 16 ightarrow 37 ightarrow 58 ightarrow 89 ightarrow 145 ightarrow 42 ightarrow 20 ightarrow 4 \rightarrow \cdots$$

That is, every natural number has a path that reaches one of the components in figure 1. See Ross Honsberger’s *Ingenuity in Mathematics* (Random House, 1970, pages 83–84) for a proof of this fact.

A number is called *happy* if it eventually reaches 1 and *unhappy* otherwise.

In this article we explore happiness in several different numeral systems. We include exercises along the way that we hope will enhance your experience. The proofs-based exercises are meant to extend the ideas we present and are completely optional. We also include a few example-driven exercises that we strongly encourage you to complete, as they will be important later in the article.

**Everyone Can Find Happiness**

As we have seen, only some numbers are happy. Maybe this seems unfair. However, the definition of happiness depends upon the method we use to represent positive integers: We rely heavily on the base-10 notation of our number system. Perhaps working in a different base would allow different numbers to be happy.

Recall that if $b \geq 2$ is an integer, then any nonnegative integer can be written uniquely as a sum

$$n_0 \cdot b^0 + n_1 \cdot b^1 + \cdots + n_k \cdot b^k,$$

where each digit $n_i$ belongs to $\{0, 1, \ldots, b-1\}$. We will express the base-$b$ representation of a number as a list
with its least significant digit written first:

$$[n_0, n_1, \ldots, n_k].$$

For example, in base 10,

$$20 = 0 \cdot 10^0 + 2 \cdot 10^1 = [0, 2]_{10},$$

and in base 2,

$$20 = 0 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 4^1 + 2 = [0, 0, 1, 0, 1]_2.$$

Consequently, we can explore the digital sum of squares for numbers represented in different bases. Let’s introduce the base-b digital sum of squares function, $S_b(n)$. So, for instance, our original base-10 function yields $S_{10}(20) = 4$, and in base 2, $S_2(20) = 0^2 + 0^2 + 1^2 + 2^1 = 2$. Notice that regardless of the base $b$, the sum of the squares of the digits is computed in base 10.

Once again, a number is base-$b$ happy if it eventually reaches 1 upon repeated application of the base-$b$ digital sum of squares function.

Let’s try this out on 20 using the base-2 digital sum of squares function:

$$20 \rightarrow 2 \rightarrow 1 \rightarrow 1$$

$$[0, 0, 1, 0, 1]_2 \rightarrow [0, 1]_2 \rightarrow [1]_2$$

So, 20 is base-2 happy, even though it is not base-10 happy (it is in the cycle containing 4). In fact, every positive integer is base-2 happy. Similarly, every number is also base-4 happy.

**Exercise:** Prove that every number is base-2 happy and base-4 happy.

**Exercise:** How many base-3 unhappy numbers can you find?

### A Rational Extension

Although it is well known that we can represent numbers in base $b$ when $b$ is an integer greater than 1, it is less well known that we can represent integers using fractional bases.

Specifically, if $p$ and $q$ are relatively prime integers and $p > q$, then any nonnegative integer has a unique representation of the form

$$n_0 \left( \frac{1}{q} \right) + n_1 \left( \frac{1}{q} \right)^2 + \cdots + n_k \left( \frac{1}{q} \right)^k,$$

where each digit $n_i$ belongs to $\{0, 1, \ldots, p - 1\}$. In particular, when $q = 1$, we recover ordinary base-$p$ representations. Again, we will write the base-$\frac{p}{q}$ representation of a number as a list $[n_0, n_1, \ldots, n_k]_{p/q}$.

This seems as if it shouldn’t work. But it does! As an example,

$$7 = 1 \cdot \left( \frac{1}{2} \right)^0 + 1 \cdot \left( \frac{3}{2} \right)^1 + 2 \cdot \left( \frac{1}{2} \right)^2 = \left[ 1, 1, 2 \right]_{3/2}.$$  

For the rest of this article, we will focus on base-$\frac{3}{2}$ representations of numbers.

**Exercise:** Take a break, and get your hands dirty! Generate the base-$\frac{3}{2}$ representations of the integers from 1 to 14. Check your answers in table 1 (where the subscript $\frac{3}{2}$ has been omitted for appearance).

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\frac{3}{2}$-representation</th>
<th>$\frac{3}{2}$-representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>[1]</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>[2]</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>[0,2]</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>[1,2]</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>[2,2]</td>
<td>12</td>
</tr>
<tr>
<td>6</td>
<td>[0,1,2]</td>
<td>13</td>
</tr>
<tr>
<td>7</td>
<td>[1,1,2]</td>
<td>14</td>
</tr>
</tbody>
</table>

**Table 1. Base-$\frac{3}{2}$ representations of 1 to 14.**
By inspecting table 1, we can glean a few interesting properties of base-$\frac{3}{2}$ representations of positive integers, all of which are true in general. First, in contrast to ordinary base-$b$ representations when $b$ is an integer, not every digit sequence yields an integer in base-$\frac{3}{2}$. For example, $[2,2,2]_{3/2} = 19/2$. More specifically, if $n = [n_0, n_1, \ldots, n_k]_{3/2} \geq 6$, then $n_k = 1$, $n_k = 2$, and $n_0$ is the remainder when we divide $n$ by 3. Moreover, $\frac{2}{3}(n - n_0)$ is an integer, and its base-$\frac{3}{2}$ representation is $[n_1, \ldots, n_k]_{3/2}$.

For example, $20 = [2,0,2,1,2]_{3/2}$. As promised, the base-$\frac{3}{2}$ representation of 20 ends with a 1 and a 2, and when we divide 20 by 3 we obtain a remainder of 2. Moreover,

$$\frac{2}{3}(20 - 2) = 12 = [0,2,1,2]_{3/2}.$$ 

**Exercise:** Prove that these properties hold for base-$\frac{3}{2}$ representations of integers greater than 5.

**Is Happiness Rational?**

Now we can play the happiness game as we did in base-$b$ by repeatedly applying the base-$\frac{3}{2}$ digital sum of squares function, which we denote $S_{3/2}$.

For example, starting with 10 gives the sequence

$10 \rightarrow 6 \rightarrow 5 \rightarrow 8 \rightarrow 9 \rightarrow [1,0,1,2]_{3/2} \rightarrow [0,1,2]_{3/2} \rightarrow [2,2]_{3/2} \rightarrow [2,1,2]_{3/2} \rightarrow [0,0,1,2]_{3/2} \rightarrow \ldots$

which then enters the cycle $5 \rightarrow 8 \rightarrow 9 \rightarrow 5$.

Let’s try a different example:

$[2,2,2,1,2]_{3/2} \rightarrow [2,1,0,1,2]_{3/2}$

Because we’ve reached 10, the previous example tells us that this sequence will eventually enter the cycle $5 \rightarrow 8 \rightarrow 9 \rightarrow 5$ as well.

**Exercise:** Gather more data. What happens to each number in table 1 when we repeatedly apply $S_{3/2}$?

In your experimentation, you should have found that every number other than 1 eventually enters the cycle $5 \rightarrow 8 \rightarrow 9 \rightarrow 5$ (see figure 2 for a visualization). Is 1 the only base-$\frac{3}{2}$ happy number? Does every number other than 1 suffer the Sisyphean fate of cycling $5 \rightarrow 8 \rightarrow 9 \rightarrow 5$ for eternity?

In doing these exercises, you may have also noticed that for $n$-values 9 through 14, $1 < S_{3/2}(n) < n$. If this inequality holds for all $n \geq 9$, then we would know that every number other than 1 eventually enters the cycle $5 \rightarrow 8 \rightarrow 9 \rightarrow 5$. Indeed, if we assume that every number $2,3,\ldots,n - 1$ eventually enters the cycle, then, because $1 < S_{3/2}(n) < n$, $S_{3/2}(n)$ eventually enters the cycle, and hence so does $n$.

Let’s prove the claim that for all $n \geq 9$, $1 < S_{3/2}(n) < n$. Let $n = [n_0, n_1, \ldots, n_k]_{3/2} \geq 9$. First, notice that $S_{3/2}(n) \geq n_0^2 = 4 > 1$.

Now we give a proof by induction that $S_{3/2}(n) < n$. You already checked this by hand for the base cases $n = 9,10,\ldots,14$. So, we may assume $n \geq 15$ and that $S_{3/2}(i) < i$, for all $9 \leq i < n$.

Remember that $\frac{2}{3}(n - n_0) = [n_1,\ldots,n_k]_{3/2}$, which, for ease of notation, we denote as $m$. So,

$$S_{3/2}(n) = n_0^2 + S_{3/2}(m).$$

Since $n \geq 15$ and $n_0 \leq 2$, we see that $m \geq \frac{2}{3}(15 - 3) = 8.6$. However, $m$ is an integer, so $m \geq 9$. Also, since $n_0 \geq 0$,

$$m = \frac{2}{3}n - \frac{2}{3}n_0 \leq \frac{2}{3}n < n.$$ 

Therefore, $9 \leq m < n$. By our inductive assumption, we know that $S_{3/2}(m) < m$.

Thus, continuing our earlier calculation,

$$S_{3/2}(n) = n_0^2 + S_{3/2}(m) < 4 + m < 4 + \frac{2}{3}n < n.$$ 

The final inequality comes from the fact that $4 < \frac{1}{3} \cdot 15 < \frac{4}{3}n$.

**Note:** If you couldn’t complete all the earlier exercises on your first pass, can you steal inspiration from the techniques presented in this section to complete those proofs?

**Further Questions**

It seems that base-$\frac{3}{2}$ is not a particularly happy base, since 1 is the only base-$\frac{3}{2}$ happy number. Regardless, it is still surprising that all other numbers eventually converge to a single cycle.
We conclude with a few open questions about happy numbers that might lead to great student research projects.

1) Two extensions of the digital sum of squares function are the digital sum function and the digital sum of cubes function, in which we take the sum of the digits or cubes of the digits of a number, respectively. More generally, Helen Grundman and Elizabeth Teeple extended this to studying the sum of dth powers of digits for any \( d \geq 1 \) (Generalized happy numbers, Fibonacci Quarterly 39 no. 5 [2001] 462–466). What happens if we apply these generalizations in base-\( \frac{3}{2} \)?

2) When \( q > 1 \), 1 is the only base-\( \frac{p}{q} \) happy number. Why? Suppose \( S'_{p/q}(n) = 1 \). Then \( n \) must have the form \([0,\ldots,0]_{p/q} = 1 \cdot \left(\frac{1}{q}\right)^k \) for some \( k \geq 0 \). But this value will be an integer only when \( k = 0 \). Thus every integer \( n \geq 2 \) is base-\( \frac{p}{q} \) unhappy. What are the possible terminal cycles?

3) For any number \( n \), can you find a base \( b > n \) (besides 2 or 4) in which \( n \) is happy?

4) Richard Guy has asked about the density of happy numbers in base-10 (see page 358 in Unsolved Problems in Number Theory, Third Edition, Springer-Verlag, 2004). Concretely, if \( H(n) \) is the number of happy numbers between 1 and \( n \), does \( \lim(H(n)/n) \) exist? If so, what is its value?

5) As we have seen, every integer enters the cycle \( 5 \to 8 \to 9 \to 5 \) under repeated application of the base-\( \frac{3}{2} \) digital sum of squares function. According to figure 2, it appears that many numbers enter this cycle at 5. In relation to the previous question, can you say anything about the density of numbers in base-\( \frac{3}{2} \) that enter the cycle at 5, 8, or 9? Moreover, are there infinitely many numbers that enter the cycle at each point?

6) The height of a base-\( b \) happy number is the minimal number of times \( S_b \) must be applied until we enter the cycle \( 5 \to 8 \to 9 \to 5 \). Are there numbers with arbitrarily large height in base-\( \frac{3}{2} \)?

We have seen that being both rational and happy is a rare phenomenon in this context. Nonetheless, the properties of the digital sum of squares function in nonintegral bases seem worthy of further study. We hope our questions have inspired you to play around with rational base representations and ask your own questions as well.

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