EXTENDING SOME FIBONACCI–LUCAS RELATIONS

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Abstract. We give a generalization of two recently proven Fibonacci-Lucas identities.

In three recent issues of *The American Mathematical Monthly* (see [3], [1], and [2]), two Fibonacci-Lucas relations were demonstrated:

$$2^{m+1}F_{m+1} = \sum_{i=0}^{m} 2^i L_i \qquad \text{and} \qquad 3^{m+1}F_{m+1} = \sum_{i=0}^{m} 3^i L_i + \sum_{i=0}^{m+1} 3^{i-1} F_i.$$

We show these two formulae are part of a family of such identities obtained by replacing 2 and 3 respectively by any integer $k \geq 1$. Using our formulation, the proof is elementary.

Theorem 0.1. *For all integers $m \geq 1$ and $k \geq 1$, we have*

$$k^{m+1}F_{m+1} = \sum_{i=0}^{m} k^i L_i + (k-2)\sum_{i=0}^{m+1} k^{i-1} F_i.$$

Proof. It is well known that $L_i + F_i = F_{i-1} + F_{i+1} + F_i = 2F_{i+1}$ so that, after rearranging sums, we get

$$\begin{aligned}
\sum_{i=0}^{m} k^i L_i + (k-2)\sum_{i=0}^{m+1} k^{i-1} F_i &= \sum_{i=0}^{m} k^i (L_i + F_i) + k^{m+1}F_{m+1} - 2\sum_{i=0}^{m+1} k^{i-1} F_i \\
&= \sum_{i=0}^{m} k^i (L_i + F_i) + k^{m+1}F_{m+1} - 2\sum_{i=0}^{m+1} k^{i-1} F_i \\
&= 2\sum_{i=0}^{m} k^i F_{i+1} + k^{m+1}F_{m+1} - 2\sum_{i=0}^{m+1} k^{i-1} F_i \\
&= k^{m+1}F_{m+1}
\end{aligned}$$

as required (since $F_0 = 0$).

□

References

[1] Harris Kwong. An alternate proof of Sury's Fibonacci-Lucas relation. *Amer. Math. Monthly*, 121(6):514, 2014.

[2] Diego Marques. A New Fibonacci–Lucas Relation. *Amer. Math. Monthly*, 122(7):683, 2015.

[3] B. Sury. A polynomial parent to a Fibonacci-Lucas relation. *Amer. Math. Monthly*, 121(3):236, 2014.

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