magician lets an audience member count out 24 playing cards from a well-shuffled deck. The volunteer chooses one of the cards (let’s say it’s the ace of spades), shows it to the audience but not the magician, reinserts it into the stack, and shuffles this 24-card deck. Another audience member then tells everyone her favorite number between 1 and 24 (let’s assume she chooses 18).

The magician alternately deals the cards into two stacks, each with 12 cards, as in figure 1, and asks the first audience member to point at the stack containing the chosen card.

After consolidating the stacks into a single deck, the magician deals the cards into three stacks, each with eight cards, as in figure 2, again asking for the volunteer to point to the stack containing the card.

The magician grabs the stacks and deals out four stacks, each with six cards, as in figure 3. For the final time, the audience member points to the stack containing the card and the magician gathers up the stacks one at a time.
Finally, the magician deals the cards out, counting them one at a time, as in figure 4, until card 18—the number chosen by the second volunteer—and voila! It is the ace of spades!

This trick, the 24-card trick, is a variation on the more famous 27-card trick, which in turn provides an interesting twist on the classic 21-card trick. These tricks have various generalizations; for instance, Ethan Bolker describes how to generate similar tricks that vary the number and size of the stacks ([2010] “Gergonne’s Card Trick, Positional Notation, and Radix Sort.” Math. Mag. 83: 46–49). We especially like this 24-card variant because the key to performing it relies on representing integers using the factorial base number system, sometimes referred to as the factoradic number system.

**The Factoradic Number System**

The factorial of a positive integer is the product of all the positive integers up to that number; so, \( m! = m(m-1)(m-2)\cdots 2\cdot 1 \). The factorial base number system allows us to write each positive integer \( n \) as a sum of factorials. More precisely, \( n \) can be written uniquely in the form

\[
n = n_i \cdot 1! + n_2 \cdot 2! + n_3 \cdot 3! + \cdots + n_k \cdot k!,
\]

where \( 0 \leq n_i \leq i \) for each index \( i \). For example, \( 17 = 1 \cdot 1! + 2 \cdot 2! + 2 \cdot 3! \) and \( 100 = 0 \cdot 1! + 2 \cdot 2! + 0 \cdot 3! + 4 \cdot 4! \). The factorials act as the positions for this representation instead of the powers of 10 as in the familiar base-10 number system. We can compactly write these expressions by listing the “digits” \( n_1, \ldots, n_k \): \( 17 = (1,2,2) \) and \( 100 = (0,2,0,4) \). Note that we write the digits from least significant to most significant. Table 1 shows the factorial base representations of the numbers 0 to 23.

Performing this magic trick depends on being able to quickly compute the factorial base representation of a number. There are two viable options to do this. We could use a greedy algorithm: find the largest factorial less than the number, subtract the largest multiple of that factorial that is less than the number, and then repeat. For instance, suppose we wish to compute the factoradic representation of 83. Because \( 4! = 24 < 83 < 5! = 120 \), we observe that \( 83 = 3 \cdot 4! + 11 \). Then, \( 11 = 1 \cdot 3! + 5 \) and \( 5 = 2 \cdot 2! + 1 \). So,

\[
83 = 1 \cdot 1! + 2 \cdot 2! + 1 \cdot 3! + 3 \cdot 4! = (1,2,1,3).
\]

A more efficient method arises by repeated use of the division algorithm. Given \( n \), find the unique values of \( q_i \) and \( r_i \) satisfying \( n = r_i + 2q_i \) where \( r_i \) is 0 or 1; then \( r_i \) gives us the least significant digit \( n_i \) for \( n \). Next, find the unique values of \( q_2 \) and \( r_2 \) satisfying \( q_1 = r_2 + 3q_2 \) with \( r_2 \) equal to 0, 1, or 2. The digit \( n_2 \) is given by \( r_2 \). Continue in this fashion finding unique \( q_i \) and \( r_i \) satisfying \( q_{i-1} = r_i + (i+1)q_i \) where \( 0 \leq r_i \leq i \) so that \( r_i \) is the digit \( n_i \). For 83, we have

\[
83 = 1 + 2 \cdot 41
\]

\[
41 = 2 + 3 \cdot 13
\]

\[
13 = 1 + 4 \cdot 3
\]

\[
3 = 3 + 5 \cdot 0.
\]

So,

\[
83 = 1 + 2(2 + 3(1 + 4(3 + 5 \cdot 0)))
\]

\[
= 1 \cdot 1! + 2 \cdot 2! + 1 \cdot 3! + 3 \cdot 4!.
\]

<table>
<thead>
<tr>
<th>Table 1. Factorial base representations.</th>
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<tbody>
<tr>
<td>Number</td>
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<tr>
<td>0</td>
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</table>
Performing the Trick
As we see in table 1, each number between 0 and 23 uses at most three factorial base digits; this corresponds to the three sets of dealings. The digits instruct the magician how to perform the trick. Given a chosen number $n$, he determines the factorial base representation of $n - 1$; let’s say that $n - 1 = (a_1, a_2, a_3)$. This representation tells him how to collect and organize the piles after each deal.

In each round, the magician picks up the piles one at a time and places them in his other hand face up in any order except that the pile identified by the audience member must be picked up $(a_i + 1)$st, or said another way, when the complete stack is face down, it will be the $(a_i + 1)$st pile from the top.

In our example, the audience member chose 18, and $17 = (1, 2, 2)$. This coding tells the magician to pick up the pile containing the ace of spades second in the first deal (figure 1), third in the second deal (figure 2), and third in the third deal (figure 3).

To give an intuition for why the trick works, we note that the first 1000 = 10³ numbers 0 to 999 fit into a $10 \times 10 \times 10$ cube; the three digits tell us in which row, column, and level to find the number (if we lay out the first 10 numbers in the first row of the bottom level and continually add 10 to create the rest of the rows to fill in 100, and then add 100 to each successive level). With this analogy in mind, the numbers 0–23 fit in a $2 \times 3 \times 4$ rectangular prism. The first row in the bottom level consists of 0 and 1, and each successive row in that level is obtained by adding 2 to the previous row. Then, each successive level is obtained by adding 6 to each entry in the previous level. See figure 5 for a schematic diagram of this prism. As we deal the piles and ask for locations, we are using the factorial base representation to give instructions for the row, column, and level where we eventually want the card to appear.

Using the Full Deck
As a bonus, we describe how to perform a similar trick using the entire deck of 52 cards. Every number between 0 and 51 can be written in the form $n_0 + n_2 10 + n_3$, where $n_0$ and $n_2$ are 0 or 1 and $0 \leq n_1 \leq 12$. (And in fact, this number system can be extended to a general mixed radix number system for the entire set of natural numbers in which the even indexed digits are 0 or 1 and the odd indexed digits are between 0 and 12.)

The magician asks for a number $n$ between 1 and 52, computes the corresponding representation of $n - 1$ in his head, and then proceeds to deal two piles, followed by 13 piles, followed by two piles, picking up the correct pile each time and placing it in the appropriate location based on $n_0$, $n_1$, and $n_2$.

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Figure 5. A $2 \times 3 \times 4$ rectangular prism. Each number from 0 to 23 would fit in its own cell. From bottom to top, the levels contain the numbers 0 to 5, 6 to 11, 12 to 17, and 18 to 23.

Solution to the puzzle on page 2