Generalized Symmetric Spaces of Dihedral Groups

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Introduction, Setup and Notation

Automorphisms and Involutions of D_n

Equivalence classes of Automorphisms of D_n

An Interesting Sequence

Description of Symmetric Spaces of D_n

The work here originated The American Institute of Mathematics.

Began as joint work with Katrina Cunningham (Southern University), Loek Helminck (North Carolina State U.), Benjamin Jones (University of Wisconsin-Stout), Hyunju Oh (Bennet College), and Rachel Schwell (Central Connecticut State), and Jennifer Vasquez (University of Scranton).

Finished with Benjamin Jones and Rachel Schwell.

Basic Setup

Let G be a group.

Let $\theta \in \mathrm{Aut}(G)$ (θ is an automorphism of G)

Important case: $\theta^2 = \mathrm{id}$ (θ is called an involution)

$H = G^\theta = \{g \in G \mid \theta(g) = g\}$ (H is called the fixed point set)

$Q = \left\{g\theta(g)^{-1} \mid g \in G\right\}$

$Q \simeq G/H$ (Q is called the symmetric space)

Classical Example: $G = \mathrm{SL}(n, \mathbb{C})$

Suppose $G = \mathrm{SL}(n, \mathbb{C}) = \{y \in M_n(\mathbb{C}) \mid \det y = 1\}$

Here's an involution of G: $\theta(A) = (A^t)^{-1}$ *"transpose inverse"*

$H = G^\theta = \{g \in G \mid \theta(g) = g\} = \mathrm{SO}(n, \mathbb{C})$ "orthogonal group"

$Q = \left\{y \in M_n(\mathbb{C}) \mid y = y^t,\ \det y = 1\right\}$ *"symmetric matrices"*

Symmetric Spaces for Dihedral Groups

All the definitions make sense for any group.

General Goals:

1. Describe the automorphisms of G up to equivalence.
2. Classify the involutions of G up to equivalence.
3. Given an automorphism (or involution), describe the symmetric space Q and the fixed point set H.
4. Determine the H-orbit structure on Q.
5. Determine the G-orbits on Q and how they split up into H-orbits.

Notation and Terminology

$\mathbb{Z}_n$ - Ring of integers mod n

$U_n = \{a \in \mathbb{Z}_n \mid ay \equiv_n 1 \text{ for some } y \in \mathbb{Z}_n\}$ - units in $\mathbb{Z}_n$

$D_n = \langle r, s \mid s^2 = 1, r^n = 1, sr = r^{n-1}s = r^{-1}s \rangle$

Each element of D_n has a "normal form"

$$D_n = \{1, r, r^2, \ldots, r^{n-1}, s, rs, r^2s, \ldots, r^{n-1}s\}$$

Note: $D_n \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2$ as groups

Note: $\text{Aut}(\mathbb{Z}_n) = U_n$

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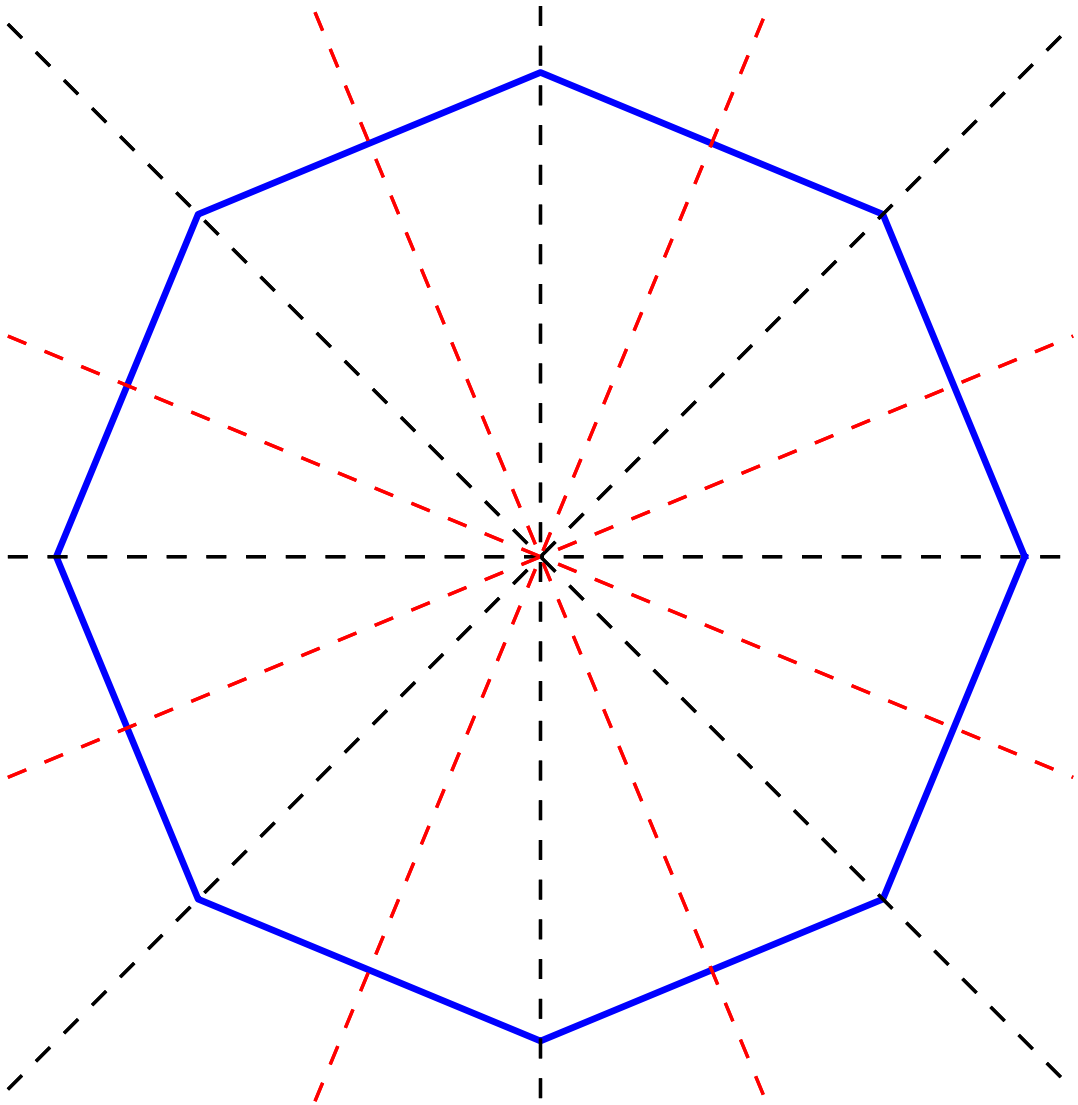
Symmetry diagram of D_8



Automorphisms and Involutions of D_n

We discovered that the hard work involved in describing symmetric spaces for D_n lies in deeply understanding the structure of the automorphism group of D_n.

We used group theory, elementary number theory, and computer experimentation using Sage[1].

[1] http://www.sagemath.org

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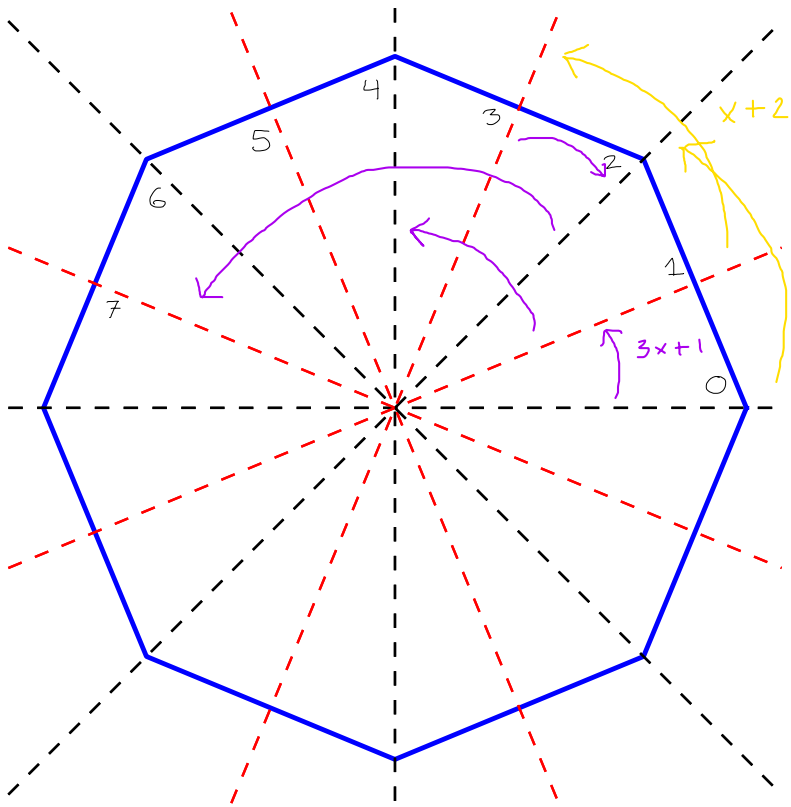
Automorphisms of D_n

$$\mathrm{Aut}(D_n) \cong \{ax + b \mid a \in U_n, b \in \mathbb{Z}_n\}$$

We view the automorphisms as acting on the reflecting planes.



Some Examples

General form: $(ax + b)(r^k s^m) = r^{ak+bm} s^m$

In particular:

conj(s)	$-x$
conj(r)	$x + 2$
Diagram automorphism	$(n-1)x + (n-1)$

Remark: If $ax + b$ is an inner automorphism then $a = \pm 1$

We say θ is *equivalent* to σ if

$$\eta\theta\eta^{-1} = \sigma \text{ for some } \eta \in \mathrm{Aut}(D_n).$$

Check: a is invariant under equivalence.

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Characterizing Automorphisms

If $\theta = ax + b$, then $\theta^k = \mathrm{id}$ if and only if

$$\begin{cases} a^k \equiv 1 & \pmod{n} \\ (a^{k-1} + a^{k-2} + \cdots + 1)b \equiv 0 & \pmod{n} \end{cases}$$

in particular: if $\theta = ax + b$ is an involution then

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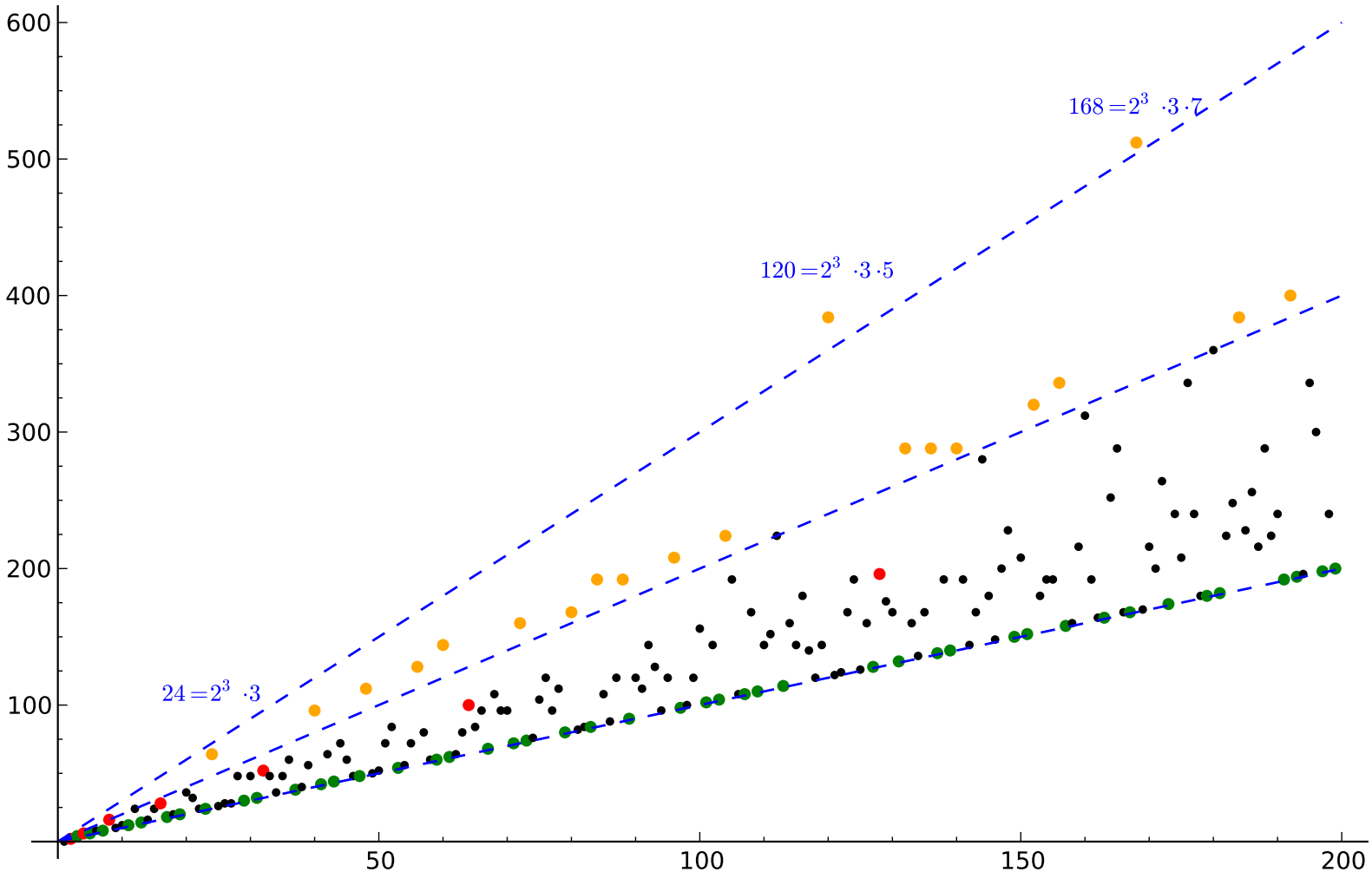
Involutions

To determine the involutions from among all automorphisms,

Solutions to $a^2 \equiv 1 \pmod{n}$ are square roots of unity mod n. Classical number theory tells us the number of these and how to construct them.

Solutions to $(a+1)b \equiv 0 \pmod{n}$ are the zero divisors of $a+1 \in \mathbb{Z}_n$; call this group $\mathrm{ZDiv}(a+1) = \left\langle \frac{n}{\gcd(a+1,n)} \right\rangle \leq \mathbb{Z}_n$.

Number of involutions of D_n vs. n



Equivalence classes of Automorphisms

Let $\mathrm{Aut}_k(D_n) = \{\theta \in \mathrm{Aut}(D_n) \mid \theta^k = \mathrm{id}\}$

Let $\mathrm{ZDiv}(c)$ be the solutions to $cx \equiv 0 \pmod{n}$.

Proposition

The set $Aut_k(D_n)$ is partitioned into equivalence classes indexed by pairs (a, B) where $a^k \equiv 1 \pmod{n}$ and B is an orbit of U_n on $\mathrm{ZDiv}(a^{k-1} + a^{k-2} + \cdots + a + 1)/\langle a - 1\rangle$.

Theorem

Let n be fixed and let $a^k \equiv 1 \pmod{n}$. Then, the number of orbits of U_n on $\mathrm{ZDiv}(a^{k-1} + a^{k-2} + \cdots + a + 1)/\langle a - 1\rangle$ is equal to the number of divisors of

$$\frac{\gcd(a-1, n)\gcd(a^{k-1} + a^{k-2} + \cdots + a + 1, n)}{n}.$$

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If $\theta = ax + b$ is an involution, then

1. $|\mathrm{ZDiv}(a+1)/\langle a-1\rangle| = \frac{\gcd(a+1,n)\gcd(a-1,n)}{n} \leq 2$
2. *We know when equality holds; equality depends only on the factorization of n and the "type" of a.*

Involutions $ax + b$ and $ax + b'$ are equivalent if $b - b' \in \langle a - 1\rangle$. and they are non-equivalent when b, b' are in different cosets of $\langle a - 1\rangle$ in $\mathrm{ZDiv}(a + 1)$.

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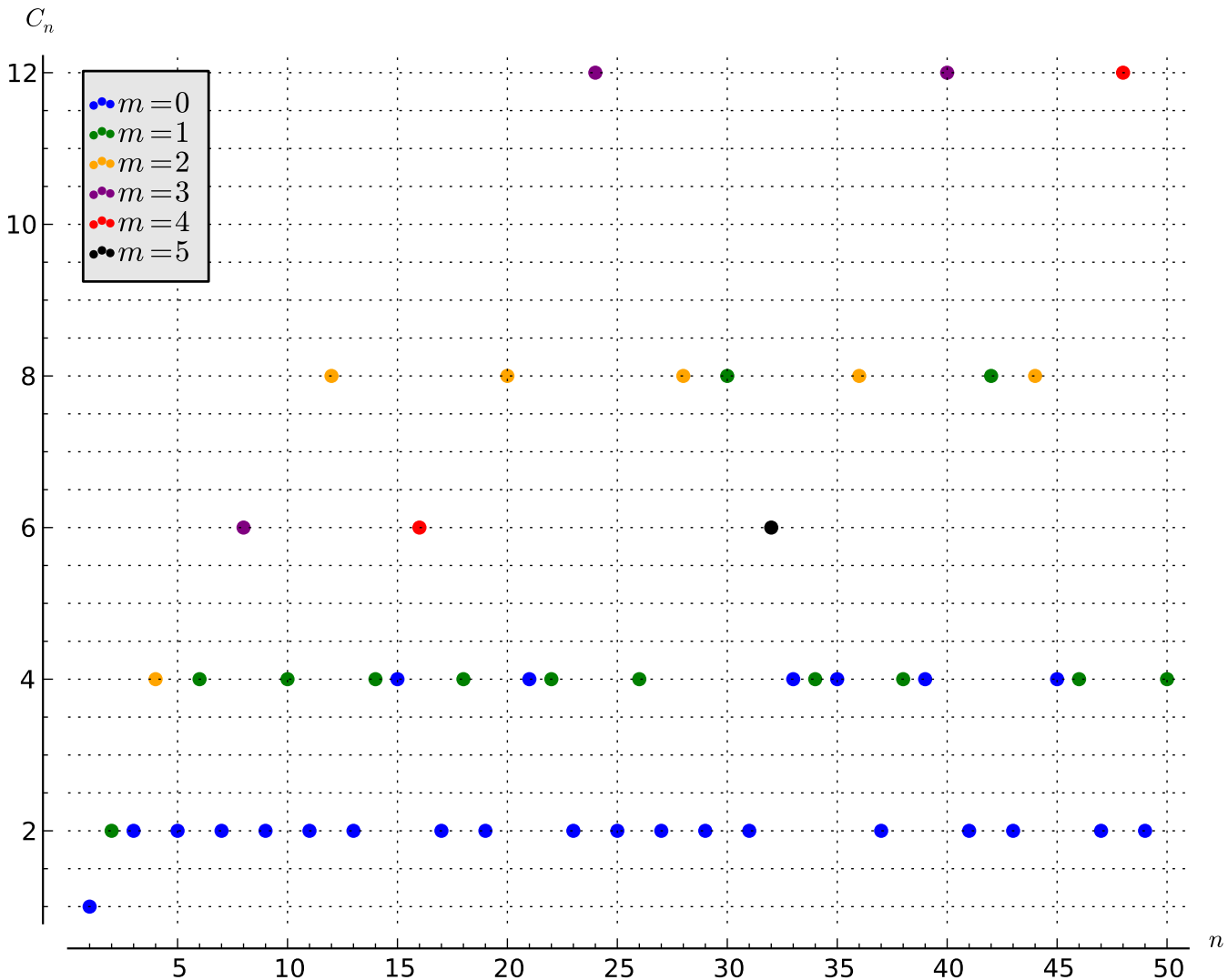
Equivalence classes of involutions

Theorem

Suppose that $n \geq 1$ *and* $n = 2^m p_1^{r_1} \cdots p_k^{r_k}$ *where the* p_i *are distinct odd primes. Then, the number of equivalence classes,* C_n*, of* $\mathrm{Invol}(D_n)$ *is given by*

$$C_n = \begin{cases} 2^{k+m} & \text{if } m < 3 \\ 2^{k+3} - 2^{k+1} & \text{if } m \geq 3. \end{cases}$$

Number of *equivalence classes* of involutions of D_n vs. n



What do you do with a sequence?

The sequence of numbers you get by counting the number of equivalence classes of involutions in D_n versus n is interesting:

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Check OEIS!

Sequence also counts the number (up to isomorphism) of groups of order $2n$ that have a subgroup isomorphic to $\mathbb{Z}_n$

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With equivalence of automorphisms understood, describing the symmetric space is relatively easy:

Theorem

Let $G = D_n$ and $\theta = ax + b \in Aut(D_n)$ be of finite order. Then

$$H = \{r^k \mid k(a-1) \equiv 0 \ (mod\ n)\} \cup \{r^k s \mid k(a-1) \equiv -b \ (mod\ n)\}$$
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Corollary

Let $\theta = ax + b$.

1. *H is either $\mathrm{ZDiv}(a-1)$ or $\mathrm{ZDiv}(a-1) \rtimes \mathbb{Z}_2$.*
2. *Q is a subgroup if $b \in \langle a-1 \rangle$ and in this case $Q \cong \langle a-1 \rangle$.*
3. *If θ is an involution Q is always a subgroup.*

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THANKS!

Any Questions?

References

On the Structure of Involutions and Symmetric Spaces of Dihedral Groups.