CONSECUTIVE FACTORIAL BASE NIVEN NUMBERS

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Abstract. We prove that there are no consecutive runs of five or more factorial base Niven numbers. Moreover, we construct an infinite family of collections of four consecutive factorial base Niven numbers.

1. Introduction

A Niven (or Harshad) number is a number that is evenly divisible by its (base 10) sum of digits. For instance, we see that 36 is a Niven number because $3 + 6 = 9$ and 36 is divisible by 9. More generally, we say a b-Niven number is a number that is evenly divisible by its base-b sum of digits. See A005349, AA049445, AA064150, and A064438 in [1] for a various sequences of base-b Niven numbers. Cooper and Kennedy [3] proved that there does not exist a run of 21 consecutive Niven numbers and they constructed an infinite family of sets of 20 consecutive Niven numbers. Grundman [4] generalized the work of Cooper and Kennedy to b-Niven numbers, proved that there does not exist a run of $2b+1$ consecutive b-Niven numbers, and conjectured that there always is a set of $2b$ consecutive b-Niven numbers. Grundman's conjecture was proved by Wilson [8] who also provided reasonable bounds on the smallest set of $2b$ consecutive b-Niven numbers (see [7] and [2] as well).

The definition of Niven numbers extends to other methods of representing integers. For instance, Cooper and Ray [6] showed that the set of Zeckendorf-Niven numbers has asymptotic density 0; Zeckendorf-Niven numbers are those divisible by their sum of digits when written in the Zeckendorf representation where a number is represented as a sum of distinct, non-consecutive Fibonacci numbers (see A014417 in [1]). Furthermore, Grundman [5] also showed that there are at most four consecutive Zeckendorf-Niven numbers and there are infinitely many such sets of numbers.

It is well known that each positive integer n can be written uniquely in the form

$$n = n_1 \cdot 1! + n_2 \cdot 2! + \cdots + n_k \cdot k! = \sum_{i=1}^{k} n_i \cdot i!$$

where $0 \leq n_i \leq i$ for all i and $n_k \neq 0$. We denote this by $n = (n_1, n_2, \ldots, n_k)_!$ (note we list the digits from least significant to most significant) and call this the *factorial base representation of* n. We can then consider numbers divisible by the sum of digits of the factorial base representation. In particular, for $n = (n_1, n_2, \ldots, n_k)_!$, we let $s_!(n) = \sum_{i=1}^{k} n_i$ be the factorial base sum of digits function, which allows us to generalize the notion of Niven numbers in this setting.

Definition 1. *A a factorial base Niven number (or Fiven number for short) is a number n such that $s_!(n)$ divides n.*

For example, 36 is a Fiven number since $36 = (0, 0, 2, 1)_!$ and 36 is divisible by $3 = 2 + 1 = s_!(36)$. We are not the first to generalize the notion of Niven numbers to the factorial base

9320542		58467310		283088806
11397166		72657574		479329774
29048470		84079006		485213542
29394574		101730310		499403806
40469902		178911502		528476542
40816006		200716054		530553166

FIGURE 1. Least element in a set of 4 consecutive Fiven numbers.

representation; for instance, see A118363 in [1]. With this terminology in place, we can now state our main theorem.

Theorem 2. *There does not exist a sequence of five consecutive factorial base Niven numbers.*

Thus, perhaps ironically, there is no run of five consecutive Fiven numbers. There are, however, runs of four consecutive factorial base Niven numbers; the table in Figure 1 lists the first 18 sets of four consecutive factorial base Niven numbers (where we only list the least element of the four consecutive integers). For example, we demonstrate that the first two numbers start runs of four consecutive factorial base Niven numbers in the table below.

n	factorial base representation of n	$s_!(n)$	$n/s_!(n)$
9320542	$(0,2,3,0,1,2,1,6,5,2)_!$	22	423661
9320543	$(1,2,3,0,1,2,1,6,5,2)_!$	23	405241
9320544	$(0,0,0,1,1,2,1,6,5,2)_!$	18	517808
9320545	$(1,0,0,1,1,2,1,6,5,2)_!$	19	490555
11397166	$(0,2,3,1,2,2,5,3,1,3)_!$	22	518053
11397167	$(1,2,3,1,2,2,5,3,1,3)_!$	23	495529
11397168	$(0,0,0,2,2,2,5,3,1,3)_!$	18	633176
11397169	$(1,0,0,2,2,2,5,3,1,3)_!$	19	599851

As with the results mentioned above, one might wonder if there are infinitely many collections of four consecutive factorial base Niven numbers, and we demonstrate this is in fact the case.

Theorem 3. *There are infinitely many collections of four consecutive factorial base Niven numbers.*

In the next section, we provide proofs of our two main theorems.

2. Proofs of Theorems

To prove Theorem 2, we need the following lemmas.

Lemma 4. *If $n = ar!$, for some $a, r \in \mathbb{N}$, then $n = (0, 0, \ldots, 0, n_r, n_{r+1}, \ldots, n_k)_!$. In other words, the first $r-1$ factorial base digits of n are all 0.*

Proof. This follows by the definition of the factorial base representation. □

Lemma 5. *Let $n \in \mathbb{N}$ such that $n = 6\ell+1$ or $n = 6\ell+3$ for some $\ell \in \mathbb{N}$. Then $s_!(n) = s_!(n+1)$.*

Proof. Let $n \in \mathbb{N}$. First assume that $n = 6\ell + 1$. By Lemma 4, since $6 = 3!$ we know that $n = (1, 0, n_3, n_4, \ldots, n_k)_!$. Now, we note that $n+1 = (0, 1, n_3, n_4, \ldots, n_k)_!$ since $1+1! = 2 = 2!$. Thus, by definition we have $s_!(n+1) = 1 + \sum_{i=3}^{k} n_i = s_!(n)$ as required.

Similarly, assume that $n = 6\ell + 3$. Again, by Lemma 4 and the fact that $3 = (1, 1)_!$ we see that $n = (1, 1, n_3, n_4, \ldots, n_k)_!$. Also, since $n + 1 = 6\ell + 4$ and $4 = (0, 2)_!$, we have that $n + 1 = (0, 2, n_3, n_4, \ldots, n_k)$. Thus, we see again that $s_!(n + 1) = 0 + 2 + \sum_{i=3}^{k} n_i = 1 + 1 + \sum_{i=3}^{k} n_i = s_!(n)$. □

We now provide a proof of Theorem 2.

Proof of Theorem 2. Let $n \in \mathbb{N}$. Then there exists $0 \leq i \leq 3$ such that $n + i = 6\ell + 1$ or $n + i = 6\ell + 3$ for some $\ell \in \mathbb{N}$. Note that $n + i$ and $n + i + 1$ must both be in a sequence of five consecutive integers beginning with n. By Lemma 5, $s_!(n + i) = s_!(n + i + 1)$.

Now, suppose $n+i$ and $n+i+1$ are each Fiven numbers and hence divisible by $s_!(n+i) = s_!(n + i + 1)$. Then, $(n + i + 1) - (n + i) = 1$ must be divisible by $s_!(n + i)$ as well. This can only be true when $s_!(n + i) = 1$. However, if $n + i = 6\ell + 3$, then $n + i = (1, 1, n_3, \ldots, n_k)_!$, so $s_!(n + i) \geq 2 \neq 1$. Thus, $n + i = 6\ell + 1$, so $n + i = (1, 0, n_3, n_4, \ldots, n_k)_!$. Since $s_!(n + i) = 1$, we conclude $n_3 = n_4 = \cdots = n_k = 0$, which implies that $n + i = 1$ so that $n = 0$ or $n = 1$. However, each sequence of five consecutive integers starting from n contains $3 = (1, 1)_!$, which is not a Fiven number. This contradiction shows that there does not exist a sequence of five consecutive Fiven numbers. □

To prove Theorem 3, we will show that $n! + 4821190, n! + 4821191, n! + 4821192$, and $n! + 4821193$ are all Fiven numbers when $n \geq 23$.

Proof of Theorem 3. We first note that $4821193 < 11!$ and $4821190 = (0, 2, 3, 2, 0, 4, 4, 2, 3, 1)_!$. Now, when $n \geq 23$, we have

$$\begin{aligned}
n! + 4821190 &= (0, 2, 3, 2, 0, 4, 4, 2, 3, 1, 0, \ldots, 0, 1)_! \\
n! + 4821191 &= (1, 2, 3, 2, 0, 4, 4, 2, 3, 1, 0, \ldots, 0, 1)_! \\
n! + 4821192 &= (0, 0, 0, 3, 0, 4, 4, 2, 3, 1, 0, \ldots, 0, 1)_! \\
n! + 4821193 &= (1, 0, 0, 3, 0, 4, 4, 2, 3, 1, 0, \ldots, 0, 1)_!
\end{aligned}$$

Therefore, $s_!(n! + 4821190) = 22$, $s_!(n! + 4821191) = 23$, $s_!(n! + 4821192) = 18$, and $s_!(n! + 4821193) = 19$.

Note that 4821190, 4821191, 4821192, and 4821193 are divisible by $22, 23, 18$, and 19, respectively. Finally, since $n \geq 23$, $n!$ is divisible by each of $22, 23, 18$, and 19. Thus, each of $n! + 4821190$, $n! + 4821191$, $n! + 4821192$, and $n! + 4821193$ is divisible by the sum of its factorial base digits. □

We finish with the following result describing other infinite families of sets of 2 and 3 consecutive Fiven numbers.

Proposition 6. *For all $n \geq 3$, both $n! + 2$ and $n! + 3$ are Fiven numbers, and for all $n \geq 5$ each of $2n! + 4$, $2n! + 5$, and $2n! + 6$ is a Fiven number.*

Proof. For $n \geq 3$, $n!+2$ is even and $s_!(n!+2) = s_!((0, 1, 0, \ldots, 0, 1)_!) = 2$ and $n!+3$ is divisible by 3 and $s_!(n! + 3) = s_!((1, 1, 0, \ldots, 0, 1)_!) = 3$ so the result follows. Likewise, for $n \geq 5$, we note that $2n!+4$ is divisible by 4 and $s_!(2n!+4) = s_!((0, 2, 0, \ldots, 0, 2)_!) = 4$. Moreover, $2n!+5$ is divisible by 5 and $s_!(2n! + 5) = s_!((1, 2, 0, \ldots, 0, 2)_!) = 5$. Finally, $2n! + 6$ is divisible by 3 and $s_!(2n! + 6) = s_!((0, 0, 1, 0 \ldots, 0, 2)_!) = 3$, □

Thus, we leave the reader with the challenge of finding other such families of either two, three, or four consecutive Fiven numbers.

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