Coxeter Groups and Root Systems via Automatic Structures

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Introduction

Coxeter Groups - Important class of finitely generated groups

Finite Group Theory: Coxeter groups $\leftrightarrow$ reflection groups

- dihedral groups
- symmetric group

Representation Theory: Weyl groups

- semi-simple Lie algebras
- Lie group theory
- E_8

Combinatorics

Automatic Group Theory

Introduction

Finite State Automata

Let S be a finite set (called an alphabet).

Definition

A *finite state automaton* for the alphabet S is a tuple (A, a_0, μ, A_f) where

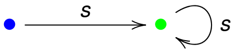
- A is a finite set (called the *states* of the automaton)
- $a_0 \in A$ (called the *initial state*)
- $\mu : S \times A \to A$ (called the *transition function*)
- $A_f \subseteq A$ (called the *final states* or *accept states*)

Examples

Example

$S = \{s\}$ $\quad A = \{{\color{blue}\bullet}, {\color{green}\bullet}\}$ $\quad a_0 = {\color{blue}\bullet}$ $\quad A_f = \{{\color{green}\bullet}\}$

μ is best described by the edges of a graph (shown below):



Remarks:

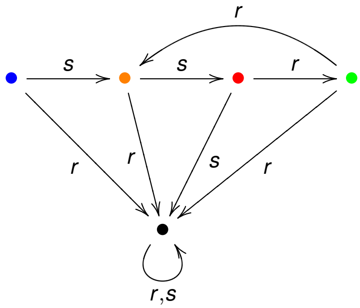
1. Vertices $\longleftrightarrow A$. Each vertex should have one originating, directed, labeled edge for each element of S.
2. We often write $s \cdot a$ instead of $\mu(s, a)$.

Examples

Example

$S = \{r, s\}$ $A = \{\bullet, \bullet, \bullet, \bullet, \bullet\}$ $a_0 = \bullet$ $A_f = \{\bullet, \bullet\}$

Again, we describe μ by a graph:



Regular Languages

Let S be a finite alphabet and $\mathcal{A} = (A, a_0, \mu, A_f)$.

1. S^* is the set of all words on S (including the empty word).
2. $L \subseteq S^*$ is called a language.
3. We can easily extend μ so that $\mu : S^* \times A \to A$.
4. Let $w = s_1 ... s_n \in S^*$. We say $\mathcal{A}$ accepts w if

$$w \cdot a_0 := \mu(w, a_0) \in A_f.$$

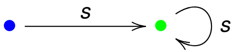
5. Let $L(\mathcal{A}) \subseteq S^*$ to be the collection of words accepted by $\mathcal{A}$.
6. A language $L \subseteq S^*$ is *regular* over S if $L = L(\mathcal{A})$ for some finite state automaton $\mathcal{A}$.

Examples, Revisited

Example

$S = \{s\}$ $A = \{\bullet, \bullet\}$ $a_0 = \bullet$ $A_f = \{\bullet\}$

μ from graph:



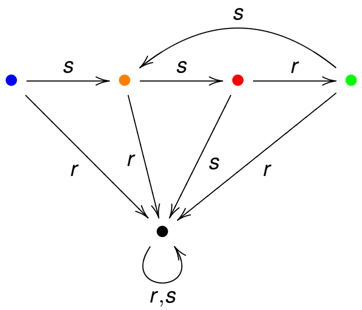
$$L(\mathcal{A}) = \{s^m \mid m \geq 1\} = S^* \setminus \{\epsilon\}$$

Examples

Example

$S = \{r, s\}$ $\quad A = \{{\color{blue}\bullet}, {\color{green}\bullet}, {\color{orange}\bullet}, {\color{red}\bullet}, \bullet\}$ $\quad a_0 = {\color{blue}\bullet}$ $\quad A_f = \{{\color{green}\bullet}, {\color{red}\bullet}\}$

μ from graph:



$$L(\mathcal{A}) = \{(ssr)^m \mid m \geq 1\} \cup \{(ssr)^m ss \mid m \geq 0\}$$

More Examples

Example

Let $S = \{r, s\}$. Define $L = \{(sr)^n \mid n \in \mathbb{N}\}$. Then L is regular.

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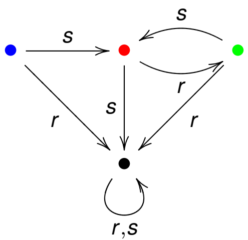
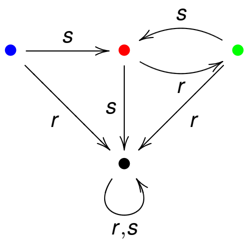


More Examples

Example

Let $S = \{r, s\}$. Define $L = \{(sr)^n \mid n \in \mathbb{N}\}$. Then L is regular.



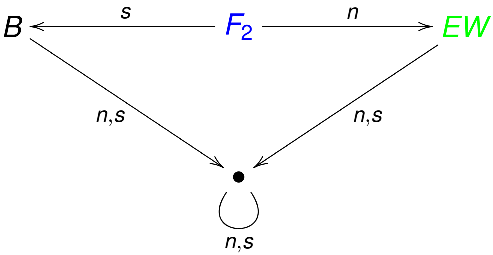
Example

$S = \{r, s\}$. Define $L = \{s^n r^n \mid n \in \mathbb{N}\}$. Then L is not regular.

Groundhog Day

Let $S = \{\text{s, n}\}$ and $A = \{F_2, EW, B\}$.

Then if $a_0 = F_2$ and $A_f = \{EW\}$ and μ is given by



Then $L(\mathcal{A}) = \{n\}$.

Point: Any finite language is regular.

Closure

Theorem

If L and K are both regular languages over S, then $L \cap K$, $L \cup K$, and L^c are all regular.

Proof Idea.

Let $\mathcal{A} = (A, a_0, \mu, A_f)$ and $\mathcal{B} = (B, b_0, \nu, B_f)$ be the FSAs for L and K respectively.

For L^c use $(A, a_0, \mu, A \setminus A_f)$

For $L \cup K$ use $(A \times B, (a_0, b_0), \mu \times \nu, (A \times B_f) \cup (A_f \times B))$

For $L \cap K$ use $(A \times B, (a_0, b_0), \mu \times \nu, A_f \times B_f)$

Note: $L \cap K = (L^c \cup K^c)^c$ by DeMorgan's Laws.

□

Finitely Generated Groups

Definition

A *group* is a set G with a operation such that

1. Operation is associative: $(ab)c = a(bc)$ for all triples
2. There is an identity element: $1g = g1 = g$ for all g
3. Every element has an inverse: $g^{-1}g = gg^{-1} = 1$.

Definition

We say G is *generated* by $S \subseteq G$ if every element of G can be written as a product of elements in $S \cup S^{-1}$. We write $G = \langle S \rangle$.

Definition

We say $G = \langle S \rangle$ is *finitely generated* if S is finite. $(G = \langle S \mid R \rangle)$.

Finitely Generated Groups

Example

$$\mathbb{Z} = \langle 1 \rangle$$

$$\mathbb{Z}_n = \langle 1 \mid \underbrace{1 + 1 + \cdots + 1}_{n} = 0 \rangle$$

Example

$$D_n = \langle r, s \mid r^n = s^2 = 1;\ rs = sr^{-1} \rangle$$

$$= \langle a, b \mid a^2 = b^2 = 1;\ (ab)^n = 1 \rangle$$

Example

$$S_n = \left\langle \{s_1, \ldots, s_{n-1}\} \;\middle|\; \begin{array}{c} s_i^2 = 1; \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}; \\ s_i s_j = s_j s_i \ \ (|i-j| \neq 1) \end{array} \right\rangle$$

Automatic Groups

Let G be a finitely generated group $G = \langle S \rangle$.

Definition

An *automatic structure* on G consists of a finite state automaton $\mathcal{A}$ over S and a finite state automata $\mathcal{M}_x$ over (S, S) for each $x \in S \cup \{\epsilon\}$, satisfying:

1. There is a surjective map $\pi : L(\mathcal{A}) \to G$.
2. $(w_1, w_2) \in L(\mathcal{M}_x)$ if and only if $w_1 x = w_2$ in G.

Remarks:

$\mathcal{A}$ is called the word acceptor.

$\mathcal{M}_x$ are called the multiplier (and equality) automata.

Automata over (S, S) can be made precise (more time).

Above, we abuse notation by letting $w_i \in S^*$ and $w_i \in G$.

Examples

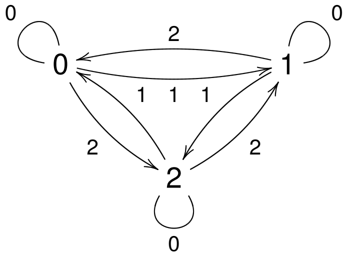
Suppose G is finite. Then $G = \langle G \rangle$ we define $\mu(g, h) = gh$.

$\mathcal{A} = (G, 1, \mu, G)$ is the word acceptor for G.

Multipliers are obtained by modifying $\mathcal{A}$.

Example

Let $G = \mathbb{Z}_3$

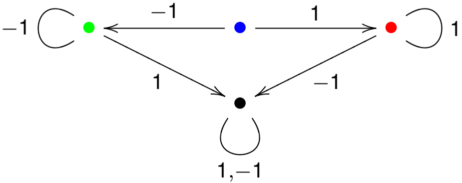


Note: In general, for $S \subseteq G$, you can use the Cayley graph for (G, S).

Examples

Example (Integers)

$\mathbb{Z} = \langle\{1, -1\}\rangle$. Let $A = \{$blue$, $green$, red, $black$\}$, $a_0 =$ blue, $A_f = \{$blue, green, red$\}$ and suppose that μ is described by the graph



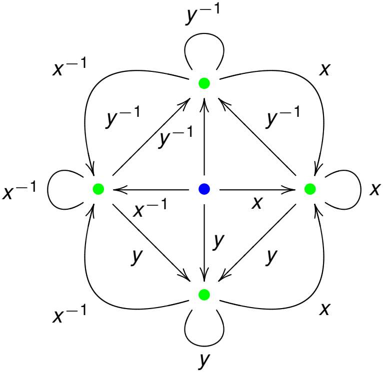
This recognizes *reduced expressions* of integers.

Multiplier automata can be built from this (length).

Examples

Example (Free Group)

Let $F_2 = \langle\{x, y, x^{-1}, y^{-1}\}\rangle$.



Missing edges go to a fail state.

Word acceptor shown, can be modified to multipliers.

Coxeter Groups

Let $S = \{s_1, ..., s_n\}$.

Definition

An $n \times n$ *Coxeter matrix* is a matrix m satisfying

$$m_{i,j} = m_{j,i}$$
$$m_{i,i} = 1$$
$$m_{i,j} \in \mathbb{Z}_{\geq 2} \cup \{\infty\} \text{ for } i \neq j.$$

Given m, we let $W = \langle S \mid (s_i s_j)^{m_{i,j}} = 1 \}$

We call (W, S) a *Coxeter system*

Notes:

$$m_{i,i} = 1 \text{ means } s_i^2 = 1$$
$$m_{i,j} = 2 \text{ means } s_i s_j = s_j s_i$$
$$m_{i,j} = n \text{ means } \underbrace{s_i s_j \ldots}_{n \text{ times}} = \underbrace{s_j s_i \ldots}_{n \text{ times}}$$

Examples

Example (Dihedral Groups)

Let $S = \{a, b\}$ and $m = \begin{pmatrix} 1 & n \\ n & 1 \end{pmatrix}$. Then

$$D_n = \langle a, b \mid a^2 = b^2 = 1;\ (ab)^n = 1 \rangle$$

Remark: a and b are both reflections of n-gon.

Examples

Example (Symmetric Groups)

Let $S = \{s_1, ..., s_{n-1}\}$. Define m by letting $m_{i,i} = 1$ and for $i \neq j$

$$m_{i,j} = \begin{cases} 2 & |i-j| \neq 1 \\ 3 & \text{otherwise} \end{cases}.$$

Then

$$S_n = \left\langle s_1, ..., s_{n-1} \,\middle|\, \begin{matrix} s_i^2 = 1; \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}; \\ s_i s_j = s_j s_i \ (|i-j| \neq 1) \end{matrix} \right\rangle$$

Remark: $s_i = (i, i+1)$ is a simple transpositions.

Examples

Example (Infinite Case)

Let $S = \{r, s, u\}$ and define the Coxeter matrix by

$$m = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 1 & \infty \\ 4 & \infty & 1 \end{pmatrix}$$

Then

$$W = \langle r, s, u \mid r^2 = s^2 = u^2 = 1;\ rsr = srs;\ ruru = urur \rangle$$

Remark: ∞ means there is no relation.

Facts and Terminology

Every element of S^* gives element in W.

There is a length function $\ell : W \to \mathbb{N}$.

$w = s_{i_1} \cdots s_{i_r}$ is a *reduced expression* for w if $\ell(w) = r$.

i.e. D_4, $w = ababbaabab = abaaabab = ababab$

so $\ell(w) = 6$ and *ababab* is a reduced expression.

There is a classification of the finite Coxeter groups by matrix

Four infinite families (A, B, D, I_2)

Six sporadic

Coxeter Groups are Automatic

How can we show that Coxeter groups are automatic?

- i.e. infinite Coxeter groups

Develop some combinatorial/geometric data

Use the data to build a word acceptor

- Recognize reduced expressions
- Find one that recognizes some unique reduced expression
- Modify to build the multiplier automata

What good does this do us?

Root Systems

Fix a Coxeter system (W, S) with $|S| = n$.

Let V be an n-dimensional $\mathbb{R}$-vector space

Choose an S-indexed basis $\Pi := \{\alpha_s \mid s \in S\}$ for V

Define bilinear form, $(- \mid -)$, on V by

$$(\alpha_{s_i} \mid \alpha_{s_j}) = \begin{cases} -\cos\left(\frac{\pi}{m_{i,j}}\right) & m_{i,j} \neq \infty \\ b_{i,j} \leq -1 & \text{o.w.} \end{cases}$$

For each $s \in S$, $\sigma_s : V \to V$ given by

$$\sigma_s(\beta) = \beta - 2(\alpha_s \mid \beta)\alpha_s$$

Extend to action of W on V: $w \mapsto \sigma_w \in End(V)$

We call this the standard reflection representation.

Root Systems

Abuse notation, for $\beta \in V$, $w(\beta) := \sigma_w(\beta)$

Definition

The root system of (W, S) is $\Phi = \{w(\alpha_s) \mid w \in W;\ \alpha_s \in \Pi\}$.

$$\Phi^+ := \{\beta \in \Phi \mid \beta = \sum_{s_i \in S} c_i \alpha_{s_i};\ c_i \geq 0\}$$

$$\Phi^- := \{\beta \in \Phi \mid \beta = \sum_{s_i \in S} c_i \alpha_{s_i};\ c_i \leq 0\}$$

Fact: $-\Phi^+ = \Phi^-$

Fact: $\Phi = \Phi^+ \sqcup \Phi^-$

Definition

For any $w \in W$, define $\Phi_w = \{\beta \in \Phi^+ \mid w(\beta) \in \Phi^-\}$.

Fact: Let $s \in S$. Then $\ell(ws) > \ell(w)$ if and only if $\alpha_s \notin \Phi_w$

A Finite State Automaton for a Coxeter Group

Definition

Let $\alpha, \beta \in \Phi^+$. We say α *dominates* β if, for all $w \in W$, $w(\alpha) \in \Phi^-$ implies that $w(\beta) \in \Phi^-$ (write $\beta \preceq \alpha$).

Theorem

The relation $\preceq$ is a partial order on Φ^+.

Theorem

Let Σ be the minimal elements of Φ^+ with respect to $\preceq$. Then Σ is finite.

Definition

For any $w \in W$, we define $D_\Sigma(w) = \Phi_w \cap \Sigma$.

Note: $A = \{D_\Sigma(w) \mid w \in W\}$ is finite.

A Finite State Automaton for a Coxeter Group

Theorem

Let $w \in W$ and $s \in S$ with $\alpha_s \notin D_\Sigma(w)$. Then

$$D_\Sigma(ws) = \{\alpha_s\} \cup \big(s(D_\Sigma(w)) \cap \Sigma\big)$$

Theorem

Coxeter groups are automatic.

Proof.

Let $A_f = \{D_\Sigma(w) \mid w \in W\}$, $A = A_f \cup \{F\}$, $a_0 = D_\Sigma(1)$ and

$$\mu(s, B) = \begin{cases} \{\alpha_s\} \cup (s(B) \cap \Sigma) & B \in A_f;\ \alpha_s \notin B \\ F & B \in A_f; \alpha_s \in B \\ F & B = F \end{cases}$$

□

Regular Subsets

We can generalize the set Σ to

$$\Sigma_n = \{\beta \in \Phi^+ \mid \beta \text{ dominates } n \text{ positive roots}\}$$

Theorem

For every n, Σ_n is finite.

This creates a large family of automata for a Coxeter group

- $D_{\Sigma_n}(w) = \Phi_w \cap \Sigma_n$.
- Fix k, can recognize words with k-reductions.
- Important subsets of Coxeter groups.

Regular Subsets

Examples of Regular Subsets

1. For $J \subseteq S$ we get W_J (standard parabolic subgroup)
2. W' reflection subgroup, $W' \backslash W / W_J$ (minimal coset reps)
3. For $\alpha, \beta \in \Phi$, $\{w \in W \mid w(\alpha) = \beta\}$
4. For $K \subseteq S$, $\{w \in W \mid \ell(wr) < \ell(w) \text{ for all } r \in K\}$
5. For $t = utu^{-1}$ be a reflection

   $\{w \in W \mid \ell(tw) < \ell(w)\}$ (half space)

   $C_W(t)$ (centralizer in W)
6. Many others including finite unions, intersections, and complements.

Poincaré Series

Definition

For a subset $U \subseteq W$ we define the Poincaré series of U to be

$$P(U; W) = \sum_{w \in U} q^{\ell(w)}.$$

Theorem

If U is a regular subset of W, then $P(U; W)$ has a rational expression.

Remark: Allows us to effectively count elements of U of a fixed length.

Open Questions in Coxeter groups

1. How big is Σ (or Σ_n)?
2. Can you effectively construct and use these automata?
3. Which subgroups regular (in particular are reflection subgroups regular)?
4. What are the representation-theoretic implications? (Lusztig's **a**-function, etc.)

Biautomatic Groups and Associated Problems

Definition

A group is *biautomatic* if it is automatic and has a family of left multiplier automata as well.

Automatic groups have solvable word problem.

Biautomatic groups have solvable conjugacy problem.

Questions:

1. Are Coxeter groups biautomatic?
2. Is there an automatic group that is not biautomatic?
3. Does solvable conjugacy problem imply biautomaticity? (word problem $\not\Rightarrow$ automatic)
4. Do automatic groups have solvable conjugacy problem?

Artin Groups

Definition

Let S be a finite set and m be a Coxeter matrix. Define

$$W = \langle S \mid \underbrace{s_i s_j \ldots}_{m_{i,j}} = \underbrace{s_j s_i \ldots}_{m_{i,j}} \text{ when } i \neq j \}.$$

W is called an *Artin group*.

Questions:

1. Are all Artin groups automatic?
2. Are all Artin groups biautomatic?

Questions

Thanks!

REFERENCES

Epstein, D, et al. *Word Processing in Groups*.

Humphreys, J. *Reflection Groups and Coxeter Groups*.

Björner and Brenti. *Combinatorics of Coxeter Groups*.

Edgar, T. Dominance and Regularity in Coxeter Groups.

Regular Subset Example

Let $W = \tilde{B}_2$, given by $m = \begin{pmatrix} 1 & 4 & 2 \\ 4 & 1 & 4 \\ 2 & 4 & 1 \end{pmatrix}$