Coxeter Groups and Root Systems via Automatic Structures

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Coxeter Groups - Important class of finitely generated groups

Finite Group Theory: Coxeter groups ↔ reflection groups
- dihedral groups
- symmetric group

Representation Theory: Weyl groups
- semi-simple Lie algebras
- Lie group theory
- $E_8$

Combinatorics

Automatic Group Theory
Introduction

Finite State Automata and Regular Languages

Automatic Groups - Definitions and Examples

Coxeter Groups, Root Systems, and some Automata

Deeper Regularity Properties and Consequences

Further Questions and Open Problems
Let $S$ be a finite set (called an alphabet).

**Definition**

A *finite state automaton* for the alphabet $S$ is a tuple $(A, a_0, \mu, A_f)$ where

- $A$ is a finite set (called the *states* of the automaton)
- $a_0 \in A$ (called the *initial state*)
- $\mu : S \times A \rightarrow A$ (called the *transition function*)
- $A_f \subseteq A$ (called the *final states* or *accept states*)
Examples

Example

$S = \{s\}$ \hspace{1cm} $A = \{\bullet, \cdot\}$ \hspace{1cm} $a_0 = \bullet$ \hspace{1cm} $A_f = \{\bullet\}$

$\mu$ is best described by the edges of a graph (shown below):

![Graph Diagram]

Remarks:

1. Vertices $\leftrightarrow A$. Each vertex should have one originating, directed, labeled edge for each element of $S$.

2. We often write $s \cdot a$ instead of $\mu(s, a)$. 
Example

\[ S = \{ r, s \} \quad A = \{ \bullet, \bullet, \bullet, \bullet, \bullet \} \quad a_0 = \bullet \quad A_f = \{ \bullet, \bullet \} \]

Again, we describe \( \mu \) by a graph:
Regular Languages

Let $S$ be a finite alphabet and $A = (A, a_0, \mu, A_f)$.

1. $S^*$ is the set of all words on $S$ (including the empty word).
2. $L \subseteq S^*$ is called a language.
3. We can easily extend $\mu$ so that $\mu : S^* \times A \to A$.
4. Let $w = s_1\ldots s_n \in S^*$. We say $A$ accepts $w$ if

$$w \cdot a_0 := \mu(w, a_0) \in A_f.$$

5. Let $L(A) \subseteq S^*$ to be the collection of words accepted by $A$.
6. A language $L \subseteq S^*$ is regular over $S$ if $L = L(A)$ for some finite state automaton $A$. 

Example

\[ S = \{s\} \quad A = \{\bullet, \bullet\} \quad a_0 = \bullet \quad A_f = \{\bullet\} \]

\( \mu \) from graph:

\[ L(A) = \{s^m \mid m \geq 1\} = S^* \setminus \{\epsilon\} \]
Examples

Example

\[ S = \{ r, s \} \quad A = \{ \bullet, \bullet, \bullet, \bullet, \bullet \} \quad a_0 = \bullet \quad A_f = \{ \bullet, \bullet \} \]

\( \mu \) from graph:

\[
L(A) = \{(ssr)^m | m \geq 1\} \cup \{(ssr)^m ss | m \geq 0\}
\]
Example

Let $S = \{r, s\}$. Define $L = \{(sr)^n \mid n \in \mathbb{N}\}$. Then $L$ is regular.
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Example

$S = \{r, s\}$. Define $L = \{s^n r^n | n \in \mathbb{N}\}$. Then $L$ is not regular.
Let $S = \{s, n\}$ and $A = \{F_2, EW, B\}$.

Then if $a_0 = F_2$ and $A_f = \{EW\}$ and $\mu$ is given by

Then $L(A) = \{n\}$.

Point: Any finite language is regular.
Closure

Theorem
If $L$ and $K$ are both regular languages over $S$, then $L \cap K$, $L \cup K$, and $L^c$ are all regular.

Proof Idea.
Let $A = (A, a_0, \mu, A_f)$ and $B = (B, b_0, \nu, B_f)$ be the FSAs for $L$ and $K$ respectively.

For $L^c$ use $(A, a_0, \mu, A \setminus A_f)$
For $L \cup K$ use $(A \times B, (a_0, b_0), \mu \times \nu, (A \times B_f) \cup (A_f \times B))$
For $L \cap K$ use $(A \times B, (a_0, b_0), \mu \times \nu, A_f \times B_f)$

Note: $L \cap K = (L^c \cup K^c)^c$ by DeMorgan’s Laws.
Finitely Generated Groups

Definition
A group is a set $G$ with a operation such that

1. Operation is associative: $(ab)c = a(bc)$ for all triples
2. There is an identity element: $1g = g1 = g$ for all $g$
3. Every element has an inverse: $g^{-1}g = gg^{-1} = 1$.

Definition
We say $G$ is generated by $S \subseteq G$ if every element of $G$ can be written as a product of elements in $S \cup S^{-1}$. We write $G = \langle S \rangle$.

Definition
We say $G = \langle S \rangle$ is finitely generated if $S$ is finite. ($G = \langle S \mid R \rangle$).
Finitely Generated Groups

Example
\[ \mathbb{Z} = \langle 1 \rangle \]
\[ \mathbb{Z}_n = \langle 1 \mid 1 + 1 + \cdots + 1 = 0 \rangle \]

Example
\[ D_n = \langle r, s \mid r^n = s^2 = 1; rs = sr^{-1} \rangle \]
\[ = \langle a, b \mid a^2 = b^2 = 1; (ab)^n = 1 \rangle \]

Example
\[ S_n = \langle \{ s_1, \ldots, s_{n-1} \} \mid s_i^2 = 1; s_is_{i+1}s_is_{i+1}s_i = s_is_{i+1}s_is_{i+1}s_i; s_is_j = s_js_i (|i - j| \neq 1) \rangle \]
Let $G$ be a finitely generated group $G = \langle S \rangle$.

**Definition**

An *automatic structure* on $G$ consists of a finite state automaton $\mathcal{A}$ over $S$ and a finite state automata $\mathcal{M}_x$ over $(S, S)$ for each $x \in S \cup \{\epsilon\}$, satisfying:

1. There is a surjective map $\pi : L(\mathcal{A}) \to G$.
2. $(w_1, w_2) \in L(\mathcal{M}_x)$ if and only if $w_1 x = w_2$ in $G$.

Remarks:

- $\mathcal{A}$ is called the word acceptor.
- $\mathcal{M}_x$ are called the multiplier (and equality) automata.
- Automata over $(S, S)$ can be made precise (more time).
- Above, we abuse notation by letting $w_i \in S^*$ and $w_i \in G$.
Examples

Suppose $G$ is finite. Then $G = \langle G \rangle$ we define $\mu(g, h) = gh$.

$\mathcal{A} = (G, 1, \mu, G)$ is the word acceptor for $G$.

Multipliers are obtained by modifying $\mathcal{A}$.

Example
Let $G = \mathbb{Z}_3$

Note: In general, for $S \subseteq G$, you can use the Cayley graph for $(G, S)$. 
Example (Integers)

\( \mathbb{Z} = \langle \{1, -1\} \rangle \). Let \( A = \{\bullet, \bullet, \bullet, \bullet\} \), \( a_0 = \bullet \), \( A_f = \{\bullet, \bullet, \bullet\} \) and suppose that \( \mu \) is described by the graph

This recognizes \textit{reduced expressions} of integers.
Multiplier automata can be built from this (length).
Example (Free Group)

Let \( F_2 = \langle \{x, y, x^{-1}, y^{-1} \} \rangle \).

Missing edges go to a fail state.

Word acceptor shown, can be modified to multipliers.
Coxeter Groups

Let \( S = \{s_1, \ldots, s_n\} \).

Definition

An \( n \times n \) Coxeter matrix is a matrix \( m \) satisfying

\[
\begin{align*}
    m_{i,j} &= m_{j,i} \\
    m_{i,i} &= 1 \\
    m_{i,j} &\in \mathbb{Z}_{\geq 2} \cup \{\infty\} \text{ for } i \neq j.
\end{align*}
\]

Given \( m \), we let \( W = \langle S \mid (s_i s_j)^{m_{i,j}} = 1 \rangle \).

We call \((W, S)\) a Coxeter system.

Notes:

\[
\begin{align*}
    m_{i,i} = 1 &\text{ means } s_i^2 = 1 \\
    m_{i,j} = 2 &\text{ means } s_i s_j = s_j s_i \\
    m_{i,j} = n &\text{ means } \underbrace{s_i s_j \ldots}_{n \text{ times}} = \underbrace{s_j s_i \ldots}_{n \text{ times}}
\end{align*}
\]
Example (Dihedral Groups)

Let $S = \{a, b\}$ and $m = \begin{pmatrix} 1 & n \\ n & 1 \end{pmatrix}$. Then

$$D_n = \langle a, b \mid a^2 = b^2 = 1; \ (ab)^n = 1 \rangle$$

Remark: $a$ and $b$ are both reflections of $n$-gon.
Examples

Example (Symmetric Groups)

Let $S = \{s_1, \ldots, s_{n-1}\}$. Define $m$ by letting $m_{i,i} = 1$ and for $i \neq j$

$$m_{i,j} = \begin{cases} 
2 & |i - j| \neq 1 \\
3 & \text{otherwise}
\end{cases}.$$

Then

$$S_n = \left\langle s_1, \ldots, s_{n-1} \left| s_i^2 = 1; s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}; s_i s_j = s_j s_i (|i - j| \neq 1) \right. \right\rangle$$

Remark: $s_i = (i, i + 1)$ is a simple transpositions.
Example (Infinite Case)

Let $S = \{r, s, u\}$ and define the Coxeter matrix by

$$m = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 1 & \infty \\ 4 & \infty & 1 \end{pmatrix}$$

Then

$$W = \langle r, s, u \mid r^2 = s^2 = u^2 = 1; \ rsr = srs; \ ruru = urur \rangle$$

Remark: $\infty$ means there is no relation.
Facts and Terminology

Every element of $S^*$ gives element in $W$.

There is a length function $\ell : W \to \mathbb{N}$.

$w = s_{i_1} \cdots s_{i_r}$ is a *reduced expression* for $w$ if $\ell(w) = r$.

i.e. $D_4$, $w = ababbaabab = abaaabab = ababab$

so $\ell(w) = 6$ and $ababab$ is a reduced expression.

There is a classification of the finite Coxeter groups by matrix

Four infinite families $(A, B, D, I_2)$

Six sporadic
How can we show that Coxeter groups are automatic?

i.e. infinite Coxeter groups

Develop some combinatorial/geometric data

Use the data to build a word acceptor

Recognize reduced expressions

Find one that recognizes some unique reduced expression

Modify to build the multiplier automata

What good does this do us?
Root Systems

Fix a Coxeter system \((W, S)\) with \(|S| = n\).

Let \(V\) be an \(n\)-dimensional \(\mathbb{R}\)-vector space.

Choose an \(S\)-indexed basis \(\Pi := \{\alpha_s \mid s \in S\}\) for \(V\)

Define bilinear form, \((- \mid -)\), on \(V\) by

\[
(\alpha_{s_i} \mid \alpha_{s_j}) = \begin{cases} 
-\cos \left( \frac{\pi}{m_{i,j}} \right) & \text{if } m_{i,j} \neq \infty \\
\ b_{i,j} \leq -1 & \text{o.w.}
\end{cases}
\]

For each \(s \in S\), \(\sigma_s : V \to V\) given by

\[
\sigma_s(\beta) = \beta - 2(\alpha_s \mid \beta)\alpha_s
\]

Extend to action of \(W\) on \(V\): \(w \mapsto \sigma_w \in \text{End}(V)\)

We call this the standard reflection representation.
**Root Systems**

Abuse notation, for $\beta \in V$, $w(\beta) := \sigma_w(\beta)$

**Definition**
The root system of $(W, S)$ is $\Phi = \{ w(\alpha_s) \mid w \in W; \alpha_s \in \Pi \}$.

$$\Phi^+ := \{ \beta \in \Phi \mid \beta = \sum_{s_i \in S} c_i \alpha_{s_i}; c_i \geq 0 \}$$

$$\Phi^- := \{ \beta \in \Phi \mid \beta = \sum_{s_i \in S} c_i \alpha_{s_i}; c_i \leq 0 \}$$

Fact: $-\Phi^+ = \Phi^-$

Fact: $\Phi = \Phi^+ \sqcup \Phi^-$

**Definition**
For any $w \in W$, define $\Phi_w = \{ \beta \in \Phi^+ \mid w(\beta) \in \Phi^- \}$.

Fact: Let $s \in S$. Then $\ell(ws) > \ell(w)$ if and only if $\alpha_s \notin \Phi_w$
**A Finite State Automaton for a Coxeter Group**

**Definition**
Let $\alpha, \beta \in \Phi^+$. We say $\alpha$ *dominates* $\beta$ if, for all $w \in W$, $w(\alpha) \in \Phi^-$ implies that $w(\beta) \in \Phi^-$ (write $\beta \preceq \alpha$).

**Theorem**
The relation $\preceq$ is a partial order on $\Phi^+$.

**Theorem**
Let $\Sigma$ be the minimal elements of $\Phi^+$ with respect to $\preceq$. Then $\Sigma$ is finite.

**Definition**
For any $w \in W$, we define $D_\Sigma(w) = \Phi_w \cap \Sigma$.

Note: $A = \{D_\Sigma(w) \mid w \in W\}$ is finite.
A Finite State Automaton for a Coxeter Group

Theorem

Let \( w \in W \) and \( s \in S \) with \( \alpha_s \not\in D_{\Sigma}(w) \). Then

\[
D_{\Sigma}(ws) = \{\alpha_s\} \cup (s(D_{\Sigma}(w)) \cap \Sigma)
\]

Theorem

Coxeter groups are automatic.

Proof.

Let \( A_f = \{D_{\Sigma}(w) \mid w \in W\} \), \( A = A_f \cup \{F\} \), \( a_0 = D_{\Sigma}(1) \) and

\[
\mu(s, B) = \begin{cases} 
\{\alpha_s\} \cup (s(B) \cap \Sigma) & B \in A_f; \ \alpha_s \not\in B \\
F & B \in A_f; \ \alpha_s \in B \\
F & B = F
\end{cases}
\]
Regular Subsets

We can generalize the set $\Sigma$ to

$$\Sigma_n = \{ \beta \in \Phi^+ \mid \beta \text{ dominates } n \text{ positive roots} \}$$

**Theorem**

*For every $n$, $\Sigma_n$ is finite.*

This creates a large family of automata for a Coxeter group

$$D_{\Sigma_n}(w) = \Phi_w \cap \Sigma_n.$$  

Fix $k$, can recognize words with $k$- reductions.

Important subsets of Coxeter groups.
Examples of Regular Subsets

1. For $J \subseteq S$ we get $W_J$ (standard parabolic subgroup)
2. $W'$ reflection subgroup, $W' \backslash W / W_J$ (minimal coset reps)
3. For $\alpha, \beta \in \Phi$, $\{ w \in W \mid w(\alpha) = \beta \}$
4. For $K \subseteq S$, $\{ w \in W \mid \ell(wr) < \ell(w) \text{ for all } r \in K \}$
5. For $t = utu^{-1}$ be a reflection
   $\{ w \in W \mid \ell(tw) < \ell(w) \}$ (half space)
   $C_W(t)$ (centralizer in $W$)
6. Many others including finite unions, intersections, and complements.
Poincaré Series

**Definition**
For a subset $U \subseteq W$ we define the Poincaré series of $U$ to be

$$P(U; W) = \sum_{w \in U} q^{\ell(w)}.$$

**Theorem**
*If $U$ is a regular subset of $W$, then $P(U; W)$ has a rational expression.*

Remark: Allows us to effectively count elements of $U$ of a fixed length.
Open Questions in Coxeter groups

1. How big is $\Sigma$ (or $\Sigma_n$)?

2. Can you effectively construct and use these automata?

3. Which subgroups regular (in particular are reflection subgroups regular)?

4. What are the representation-theoretic implications? (Lusztig’s $a$-function, etc.)
Biautomatic Groups and Associated Problems

Definition
A group is *biautomatic* if it is automatic and has a family of left multiplier automata as well.

Automatic groups have solvable word problem.

Biautomatic groups have solvable conjugacy problem.

Questions:

1. Are Coxeter groups biautomatic?
2. Is there an automatic group that is not biautomatic?
3. Does solvable conjugacy problem imply biautomaticity? (word problem $\not\Rightarrow$ automatic)
4. Do automatic groups have solvable conjugacy problem?
Artin Groups

Definition
Let $S$ be a finite set and $m$ be a Coxeter matrix. Define

$$W = \langle S \mid s_is_j\ldots = s_js_i\ldots \text{ when } i \neq j \rangle.$$

$W$ is called an Artin group.

Questions:
1. Are all Artin groups automatic?
2. Are all Artin groups biautomatic?
Questions

Thanks!

REFERENCES
Humphreys, J. *Reflection Groups and Coxeter Groups.*
Björner and Brenti. *Combinatorics of Coxeter Groups.*
Edgar, T. Dominance and Regularity in Coxeter Groups.
Regular Subset Example

Let $W = \tilde{B}_2$, given by $m = \begin{pmatrix} 1 & 4 & 2 \\ 4 & 1 & 4 \\ 2 & 4 & 1 \end{pmatrix}$