ON THE FAMILY OF AUTOMATIC NORMAL FORMS FOR COXETER GROUPS

TOM EDGAR

Abstract. Let (W, S) be a Coxeter system with S finite. We introduce a conjectural family of normal forms for W associated to automatic structures similar to those first described by Brink and Howlett in [4]. In particular, we prove that this family exists for finite and affine Weyl groups as well as for right-angled Coxeter groups. In the process, we describe a few conjectures due to Dyer about closed sets of roots and their connection to certain sets of roots that arise from the study of the dominance order on the root system.

Introduction

Let (W, S) be a Coxeter system with S finite. We recall a notion of closure on the positive roots of W that has been studied by Dyer [11]. These closures are conjecturally related to subsets of the reflections $T_{\leq m}$ $(m \in \mathbb{N})$, which are defined in terms of a statistic on the reflections known as the infinite height. This conjectural relation would lead to a new and potentially interesting normal form for elements of Coxeter groups via known special cases of other conjectures of Dyer ([12]) involving initial sections of reflection orders (see [7], [6], or [8]) and closures of sets of roots. Furthermore, the results found in [13] afford the possibility of bringing the study of twisted Bruhat orders to bear on these conjectures, which we will describe below.

The main purpose of this note is to describe the family of normal forms that would exist if the conjectures are confirmed. Moreover, we prove that this family of normal forms does in fact exist for large

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classes of well-studied Coxeter groups: finite reflection groups, affine Weyl groups, and right-angled Coxeter groups.

The paper is organized as follows. In Section 1, we introduce the necessary terminology and background for Coxeter systems. In Section 2, we present a few conjectures and results due to Dyer in order to state and prove our main results of Theorems 2.5 and 2.6. We finish by using our results to describe the family of automatic normal forms for finite, affine, and right-angled Coxeter groups that conjecturally exist for all Coxeter groups; a method for obtaining this normal form is demonstrated for the affine Weyl group of type $\tilde{A}_2$.

1. Coxeter Systems

In this note, (W, S) will always represent a finite rank Coxeter system. Recall that every Coxeter system has a standard length function $l := l_{(W,S)} : W \to \mathbb{N}$. Let $T = \cup_{w \in W} wSw^{-1}$ be the set of reflections of W. Furthermore, let $N : W \to \mathscr{P}(T)$ be the reflection cocycle defined in [10] that is given by $N(w) = \{t \in T \mid l(tw) < l(w)\}$. For a general reference about Coxeter groups, consult [1] or [16].

1.1. **Root Systems and 2-Closure.** Without loss of generality, we will assume that (W, S) is realized in its standard reflection representation on a real vector space V with Π, Φ^+, and Φ denoting the set of simple roots, positive roots, and roots respectively. Recall that the set of reflections is in bijective correspondence with the set of positive roots under the bijection $\alpha \mapsto s_\alpha : \Phi^+ \to T$. For $t \in T$, we will let $\alpha_t \in \Phi^+$ denote the unique positive root α with $s_\alpha = t$.

For any subset $\Gamma \subseteq \Phi^+$, we say that Γ is 2-closed if for all $\alpha, \beta \in \Gamma$, we have $(\mathbb{R}_{\geq 0}\alpha + \mathbb{R}_{\geq 0}\beta) \cap \Phi^+ \subseteq \Gamma$. This closure is similar to the notion of closure in the crystallographic root system of a finite Weyl group as described in [2]. Furthermore, we say a subset $\Gamma \subseteq \Phi^+$ is biclosed if Γ and $\Phi^+ \setminus \Gamma$ are both 2-closed.

Remark. Let $T_\Gamma \subseteq T$ be the subset of reflections corresponding to Γ via the canonical bijection. Then, 2-closure is the same as the following. The set T_Γ is 2-closed if for all $t_\alpha, t_\beta \in T_\Gamma$ with $W' = \langle t_\alpha, t_\beta \rangle$, we have

$\{t \in W' \cap T \mid t_\alpha \prec_R t \prec_R t_\beta\} \cup \{t \in W' \cap T \mid t_\beta \prec_R t \prec_R t_\alpha\} \subset T_\Gamma$ for all reflection orders $\prec_R$ on T (see [7], [6], or [8] for information on reflection orders).

We note that initial sections of reflection orders, defined and studied in [7], are biclosed subsets. Furthermore, under the natural bijection $\Phi^+ \leftrightarrow T$, the set of all subsets of T corresponding to biclosed subsets of Φ^+ is described in [13], where it is denoted $\hat{\mathbf{A}}_{(W,S)}$.

1.2. **Reflection Subgroups and Subsets of Roots.** We call a subgroup W' of W a *reflection subgroup* of W if it is generated by the reflections it contains, $W' = \langle W' \cap T \rangle$. It was shown by Dyer ([10]) and Deodhar ([5]) independently that any reflection subgroup is also a Coxeter system; moreover, any reflection subgroup has a canonical set of Coxeter generators $\chi(W') = \{t \in T \mid N(t) \cap W' = \{t\}\}$. Since any reflection subgroup is also a Coxeter system $(W', \chi(W'))$, we let $\Phi_{W'}$, $\Phi^+_{W'}$, $\Pi_{W'} \subseteq \Phi$ be the set of roots, positive roots, and simple roots for $(W', \chi(W'))$ sitting inside the root system for (W, S). We have that $\Phi_{W'} = \{\alpha \in \Phi \mid s_\alpha \in W'\}$. For any reflection subgroup, $W' \leq W$, we get a corresponding length function $l_{(W', \chi(W'))} : W' \to \mathbb{N}$.

We say a reflection subgroup is *dihedral* if it is generated by two distinct reflections or equivalently $|\chi(W')| = 2$. Any dihedral reflection subgroup, $W' = \langle s_\alpha, s_\beta \rangle$, is contained in a unique maximal dihedral reflection subgroup, namely $\langle s_\gamma \mid \gamma \in \Phi \cap (\mathbb{R}\alpha + \mathbb{R}\beta) \rangle$. We let

$$\mathscr{M} := \{W' < W \mid W' \text{ is a maximal dihedral reflection subgroup}\}.$$

Using this set, we get the set $\mathscr{M}_\infty := \{W' \in \mathscr{M} \mid |W'| = \infty\}$. Furthermore, for any $t \in T$, we have the set $\mathscr{M}_t := \{W' \in \mathscr{M} \mid t \in W'\}$. Thus for each $t \in T$, the previous remarks imply that

$$T \setminus \{t\} = \bigcup_{W' \in \mathscr{M}_t} ((W' \cap T) \setminus \{t\}) \tag{1.1}$$

where the union is disjoint.

In addition to the notion of closure, we have the following classes of subsets of roots. These definitions are due to Dyer, and further information about them can be found in [12]. Let $\Gamma \subseteq \Phi^+$. We say Γ

is *balanced* if for all $\alpha \in \Gamma$ and $W' \in \mathscr{M}_{s_\alpha}$, then $\beta \in \Phi^+_{W'}$ such that $l_{(W',\chi(W'))}(s_\beta) < l_{(W',\chi(W'))}(s_\alpha)$ implies that $\beta \in \Gamma$. We say Γ is *bipedal* if for all $\alpha \in \Gamma$ and $W' \in \mathscr{M}_{s_\alpha}$ with $\alpha \notin \Pi_{W'}$, $\Pi_{W'} \subset \Gamma$. Finally, we say Γ is *unipodal* if for all $\alpha \in \Gamma$ and $W' \in \mathscr{M}_{s_\alpha}$, then $\Pi_{W'} = \{\beta, \gamma\}$ implies that either $\beta \in \Gamma$ or $\gamma \in \Gamma$. The following proposition is clear from the definitions.

Proposition 1.3. *Suppose* $\Gamma \subseteq \Phi^+$.

(1) *If* Γ *is balanced, then* Γ *is bipedal.*
(2) *If* Γ *is bipedal, then* Γ *is unipodal.*
(3) *If* $\Phi^+ \setminus \Gamma$ *is closed, then* Γ *is unipodal.*
(4) *If* $\Gamma \subseteq \Phi^+$ *is bipedal and* $\Delta \subseteq \Phi^+$ *is unipodal, then* $\Gamma \cap \Delta$ *is unipodal.*

1.4. **Infinite Height on Reflections and Roots.** Much of the terminology of this subsection is described in terms of the root system in [1]. We will instead use terminology associated to the set of reflections when possible. For any reflection $t \in T$, we define the *standard height* to be $h(t) := h_{(W,S)}(t) = \frac{l(t)-1}{2}$.

It is known that h is the rank function for a partial order on T, which is studied in [1], [9] and [14]. From the formula given in (1.1), we see that for $t \in T$ we have

$$h(t) = \sum_{W' \in \mathscr{M}_t} h_{(W',\chi(W'))}(t)$$

where $h_{(W',\chi(W'))}$ is the height function for $(W', \chi(W'))$. We use this characterization of height to introduce the "∞-height" of a reflection as follows.

$$h^\infty(t) := \sum_{W' \in \mathscr{M}_t \cap \mathscr{M}_\infty} h_{(W',\chi(W'))}(t). \tag{1.2}$$

The infinite height allows us to introduce a special subset of reflections $T_0 := \{t \in T |\ h^\infty(t) = 0\}$. We call this set *the set of elementary reflections*. Using the bijection between T and Φ^+, this set is in bijective correspondence with the elementary roots defined in [4]. In that paper, Brink and Howlett also demonstrate that this set is finite and use it to create an automaton that recognizes reduced expressions of Coxeter group elements. For another description of these facts, see [1]. More recently, Dyer has shown similar sets to be finite. In particular, we define, for any $n \in \mathbb{N}$, $T_n := \{t \in T \mid h^\infty(t) = n\}$. For these sets, we have the following theorem, due to Dyer, found in [9] (another proof can also be found in [15]).

Theorem 1.5. *If (W, S) is Coxeter system with S finite, then T_n is a finite set for all $n \in \mathbb{N}$.*

In [9], these sets are used to create a family of automata for Coxeter groups that we will describe and use at the end of Section 2.

The dominance order on the positive roots, as described in [1], [3], or [4], can be transported to a partial order on the reflections via the canonical bijection between these sets. It turns out that $h^\infty(t)$ counts the number of reflections strictly dominated by t (see [9], [15] or [14]), and the set of elementary reflections is the set of minimal elements of this partial order. While we have no need to introduce the dominance order here, we note that our results have a deep connection to the dominance order and the dominance order likely plays a role in the conjectures discussed in the next section.

2. Conjectures and Main Results

For any subset $\Gamma \subseteq \Phi^+$, we let $\bar{\Gamma}$ denote the 2-closure of Γ, that is the smallest 2-closed set containing Γ. The following is conjectured by Dyer in [12].

Conjecture 2.1. If $\Gamma \subseteq \Phi^+$ is unipodal, then $\bar{\Gamma}$ is biclosed.

This would strengthen the next conjecture, also in [12], parts of which were raised in [8, Remark 2.1.4].

Conjecture 2.2. Let $A := \{\Gamma \subseteq \Phi^+ \mid \Gamma \text{ is biclosed}\}$. Then A is a complete lattice with $\bigvee_{i\in I} \Gamma_i = \overline{\bigcup_{i\in I} \Gamma_i}$ for an arbitrary family $\{\Gamma_i\}_{i\in I} \subset A$.

The last conjecture extends the known fact that W under weak order is a meet semi-lattice (see [1] for example). In fact, [12] proves the following.

Theorem 2.3. *Let $\Phi_w := \Phi^+ \cap w(-\Phi^+)$ be the set of roots corresponding to $N(w)$. If $\Gamma \subseteq \Phi_w$ for some $w \in W$ and Γ is unipodal, then $\bar{\Gamma} = \Phi_x \subseteq \Phi_w$ for some $x \in W$.*

As described in [12], the previous theorem provides a formula for the join (when defined) of elements of W in weak order. For $x, y, z \in W$, $x \vee y = z$ if and only if $\overline{\Phi_x \cup \Phi_y} = \Phi_z$. There is a similar (proven) formula for meet, which we do not give.

Now, let $T_{\leq m} = \bigcup_{n \leq m} T_n = \{t \in T \mid h^\infty(t) \leq m\}$ and let $\Phi^+_{\leq m}$ be the corresponding set of roots, i.e. $\Phi^+_{\leq m} = \{\alpha \in \Phi^+ \mid h^\infty(s_\alpha) \leq m\}$. Furthermore, let $N_m(w) = N(w) \cap T_{\leq m}$.

The normal form we will describe depends on the this final conjecture, also due to Dyer [12].

Conjecture 2.4. Let (W, S) be a Coxeter system, and let $m \in \mathbb{N}$. Then $\Phi^+_{\leq m}$ is balanced.

In fact, we only need the weaker conjecture that $\Phi^+_{\leq m}$ being bipedal (see Proposition 1.3). We then have the following theorem.

Theorem 2.5. *Let (W, S) be a Coxeter system with S finite, and suppose $m \in \mathbb{N}$ such that $\Phi^+_{\leq m}$ is bipedal.*

(1) *For any $x \in W$, there is a unique $x' \in W$ such that $N_m(x') = N_m(x)$, and any $y \in W$ with $N_m(y) \supseteq N_m(x)$ can be written $y = x'y''$ with $l(y) = l(x') + l(y'')$.*
(2) *Any $1 \neq w \in W$ can be uniquely written as $w = w_1 \cdots w_n$ for certain $w_i \in W \setminus \{1\}$ with $i = 1, ..., n$ and $n \geq 1$ such that $l(w) = l(w_1) + \cdots + l(w_n)$ and $w_i = (w_i \cdots w_n)'$ for each i where $x \mapsto x'$ is as in part (1).*

Proof. Let $x \in W$. Then Φ_x is biclosed and thus unipodal by part 3 of Proposition 1.3. By assumption, $\Phi^+_{\leq m}$ is bipedal and so $\Phi_x \cap \Phi^+_{\leq m}$ is unipodal by part 4 of Proposition 1.3. Therefore, according to Theorem 2.3, we see that $\overline{\Phi_x \cap \Phi^+_{\leq m}} = \Phi_{x'}$ for some $x' \in W$. If $y \in W$ with $N_m(y) \supseteq N_m(x)$, then $\Phi_x \cap \Phi^+_{\leq m} \subseteq \Phi_y \cap \Phi^+_{\leq m} \subseteq \Phi_y$. Therefore we get that

$$\Phi_{x'} = \overline{\Phi_x \cap \Phi^+_{\leq m}} \subseteq \overline{\Phi_y} = \Phi_y,$$

and so $y = x'y''$ with $l(y) = l(x') + l(y'')$ due to well known properties of the weak order on (W, S), which can be found in [1]. This proves part (1), and part (2) follows from (1) inductively. $\square$

While it is not known for general Coxeter systems whether or not $\Phi^+_{\leq m}$ is bipedal, we will show that the theorem does hold for three large classes of interesting Coxeter systems.

Theorem 2.6. *Let (W, S) be a Coxeter system, and let $m \in \mathbb{N}$.*

(1) *If (W, S) is finite or affine, then $\Phi^+_{\leq m}$ is bipedal.*
(2) *If (W, S) is right-angled, that is $m(s, s') \in \{1, 2, \infty\}$ for all $(s, s') \in S \times S$, then $\Phi^+_{\leq m}$ is bipedal.*

Proof. If (W, S) is finite, then $\Phi^+_{\leq m} = \Phi^+$ for all $m \in \mathbb{N}$, and so the result is trivial. Suppose that (W, S) is affine. Then there is a well-known description of the standard root system for (W, S) in terms of the root system, Ψ, of the associated finite Weyl group, (W°, S°). We remind the reader of this now (see [17]). We let U denote the span of Ψ, and let Δ be the simple roots for Ψ. We define $V := U + \mathbb{R}\delta$ to be a vector space with U as a subspace of codimension one. Then, we have $\Phi = \{\alpha + n\delta |\ \alpha \in \Psi;\ n \in \mathbb{Z}\}$,

$$\Phi^+ := \{\alpha + n\delta |\ \alpha \in \Psi^+;\ n \geq 0\} \cup \{-\alpha + n\delta |\ \alpha \in \Psi^+;\ n \geq 1\},$$

and $\Pi = \Delta \cup \{\delta - \widetilde{\alpha}\}$ is the set of simple roots for Φ where $\widetilde{\alpha}$ is the highest root for Ψ. It is well known that every reflection, $s_{\alpha+n\delta}$ ($\alpha \in \Psi$), is contained in a unique infinite maximal dihedral reflection subgroup, namely the group generated by $\{s_\alpha, s_{\delta-\alpha}\}$. It follows from inspection in the dihedral case then that for $\alpha \in \Psi^+$ we get $h^\infty(s_{\alpha+n\delta}) = h^\infty(s_{-\alpha+(n+1)\delta}) = n$. Now, let $m \in \mathbb{N}$, and suppose $\alpha + e\delta \in \Phi_{\leq m}$. Let $s_{\alpha+e\delta} \in W'$ be a dihedral reflection subgroup with $\Pi_{W'} = \{\beta + n\delta, \gamma + k\delta\}$. Then, we have $\alpha + e\delta = a(\beta + n\delta) + b(\gamma + k\delta)$ with $a, b \in \mathbb{Z}_{>0}$. Therefore, $e = an + bk$, and so $an + bk \leq m$ with $a, b \in \mathbb{Z}_{>0}$. This implies that $n \leq m$ and $k \leq m$, and thus $h^\infty(s_{\beta+n\delta}) \leq n \leq m$ and $h^\infty(s_{\gamma+k\delta}) \leq k \leq m$. Hence $\Pi_{W'} \subset \Phi_{\leq m}$ as required. Since the result holds for this particular choice of root system, we can transfer to the statement to the set of reflections. Thus, the theorem holds for any choice of root system of (W, S).

To show (2), suppose that (W, S) is right-angled. We know that every dihedral reflection subgroup either has order 4 or ∞. Then for any $t \in T$ and $W' \in \mathscr{M}_t$ with $|W'| = 4$, we must have $t \in \chi(W')$ and so $h_{(W',\chi(W'))}(t) = 0$. Therefore, we must have

$$h^\infty(t) = \sum_{\substack{W' \in \mathscr{M}_t \\ |W'| = \infty}} h_{(W',\chi(W'))}(t) = \sum_{W' \in \mathscr{M}_t} h_{(W',\chi(W'))} = h(t).$$

Now suppose $\alpha \in \Phi^+_{\leq m}$ and $W' \in \mathscr{M}_{s_\alpha}$. Let $\{\beta, \gamma\} = \Pi_{W'}$. Then, by previous statement and the fact that $s_\beta \leq s_\alpha$ in Bruhat order, we get that

$$h^\infty(s_\beta) = h(s_\beta) \leq h(s_\alpha) = h^\infty(s_\alpha) \leq m,$$

and so $\beta \in \Phi^+_{\leq m}$, and similarly for γ as required. $\square$

By combining our two main results, we have the following.

Corollary 2.7. *Let $m \in \mathbb{N}$ and (W, S) be finite, affine, or right-angled. Then there exists a normal form for elements of W given by part (2) of Theorem 2.5.*

In fact, if Conjecture 2.4 holds, then this normal form exists for every (finite rank) Coxeter system. We call this normal for the m-automatic normal form for reasons we will now describe. In [9] (and generalized in [14]), a family of automata known as the m-canonical automata ($m \in \mathbb{N}$) is defined, which generalize the automaton of Brink and Howlett [4]. For fixed $m \in \mathbb{N}$, the set of states is given by

$$\{(k, N_k(x)) \mid k \leq m, x \in W\} \cup \{\infty\},$$

where ∞ is the unique "fail state." For $s \in S$, the transition function for s is given by

$$s\cdot(k, N_k(x)) = \begin{cases} \infty & \text{if } s \in N_k(x) \text{ and } k = 0 \\ (k-1, (sN_k(x)s \setminus \{s\}) \cap T_{\leq k-1}) & \text{if } s \in N_k(x) \text{ and } k > 0 \\ (k, (sN_k(x)s \cup \{s\}) \cap T_{\leq k}) & \text{if } s \notin N_k(x). \end{cases}$$

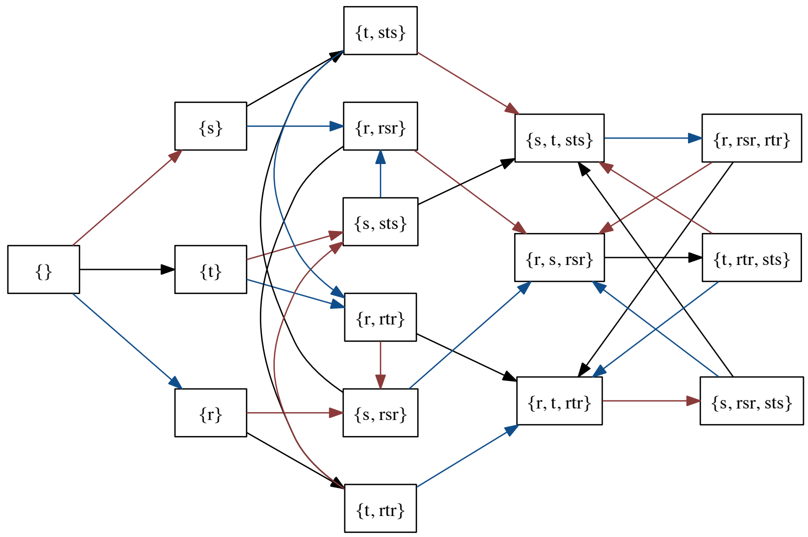

FIGURE 1. This is the 0-canonical automaton for $\tilde{A}_2$. Each state is labeled by a set of the form $N_0(w)$ for some $w \in W$; the blue edges correspond to the transition function for r, the red edges for s, and the black edges for t. Any edges not pictured lead to the (not pictured) unique "fail state," ∞, which would only be attained by using a non-reduced expression for an element of W.

Now, if we input a reduced expression for $w \in W$ into this automaton starting at $(m, N_m(1))$, we will end on the state corresponding to $(m, N_m(w))$. According to Theorem 2.5, as long as $\Phi_{\leq m}$ is bipedal there is a unique minimal element $w_1 \in W$ with $(m, N_m(w_1)) = (m, N_m(w))$. Next, we repeat this process with the shorter element $w_1^{-1}w$ and continue until we get to the identity element. In other words, the theorem guarantees a finite generating set for W consisting of the minimal length elements of W corresponding to sets of the form $N_m(w)$; every reduced word in W can be written uniquely as a product of these generators (where lengths add) by the process described.

To make the previous paragraph clear, we have the following example of the 0-automatic normal form for the Coxeter system of type $\tilde{A}_2$.

Example 2.8. Let (W, S) be the Coxeter system given by $S = \{r, s, t\}$ and

$$W = \langle S \mid r^2 = s^2 = t^2 = 1;\ (rs)^3 = (rt)^3 = (st)^3 = 1\rangle.$$

Then, we have $T_0 = \{r, s, t, rsr, rtr, sts\}$. Moreover, in this case, the 0-canonical automaton is the automaton defined by Brink and Howlett. According to [1], there are 16 states (not including the fail state) in this automaton, which we have drawn in Figure 1.

We can then use the automaton in Figure 1 to determine that the set of minimal words is

$$\{1, r, s, t, rs, rt, sr, st, tr, ts, rsr, rtr, sts, trsr, srtr, rsts\}.$$

Finally, if we take an arbitrary (reduced) word, say $w = rtstrst$, we can run the word through the automaton (from right to left) and we land on the node $\{r, rsr, rtr\}$, which corresponds to the minimal word $rsts$. We repeat this process with $(rsts)^{-1}(rtstrst) = rst$ and land on the node $\{r, rsr\}$, which corresponds to the minimal word rs. Finally, we repeat with $(rs)^{-1}(rst) = t$ and land on the node $\{t\}$, which corresponds to the minimal word t. Thus, we have the normal form $rtstrst = (rsts)(rs)(t)$. We can see that $l(rtstrst) = 7 = 4 + 2 + 1 = l(rsts) + l(rs) + l(t)$.

Remark. We finish by noting that in [14], we construct a similar family of automata associated to *generalized length functions* of Coxeter systems. We prove there that the analogs of the sets $\Phi_{\leq m}$ for the generalized length (and corresponding height) functions are not bipedal, and thus these automata do not lead to a more refined normal form.

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Department of Mathematics, Pacific Lutheran University, Tacoma, WA 98447

E-mail address: `edgartj@plu.edu`