Proof Without Words: Abel's Transformation

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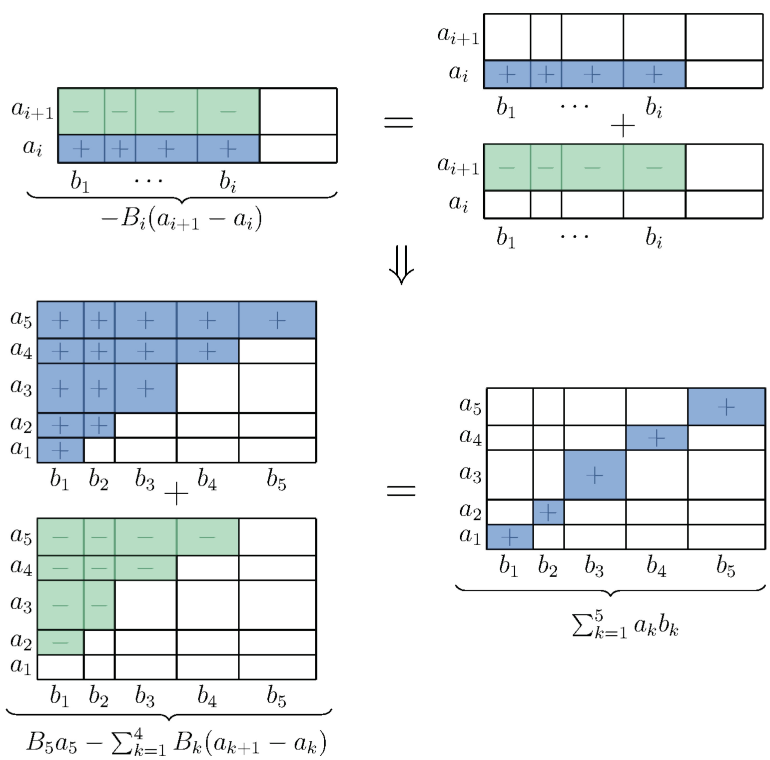
Theorem. *Let* $(a_1, a_2, a_3, \ldots)$ *and* $(b_1, b_2, b_3, \ldots)$ *be sequences of positive real numbers and let* $n \in \mathbb{N}$*. If* $B_i := b_1 + b_2 + \cdots + b_i$*, then*

$$\sum_{k=1}^{n} a_k b_k = B_n a_n - \sum_{k=1}^{n-1} B_k (a_{k+1} - a_k).$$

Proof. (e.g. for $n = 5$)



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Exercise. Use $a = (1, 2, 3, 4, \ldots)$ and $a' = (1, 4, 9, 16, \ldots)$ separately in conjunction with $b = (1, 1, 1, 1, \ldots)$ to show that, for all n,

$$\sum_{k=1}^{n} k = \frac{n^2 + n}{2} \qquad \text{and} \qquad \sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{6}.$$

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Exercise. Use Abel's transformation to provide closed formulas for $\sum_{k=1}^{n} k^3$ and $\sum_{k=1}^{n} k^4$.

Summary. We provide a wordless computation demonstrating the discrete analog of integration by parts, which is often referred to as summation by parts or Abel's transformation. We mention a few consequences.

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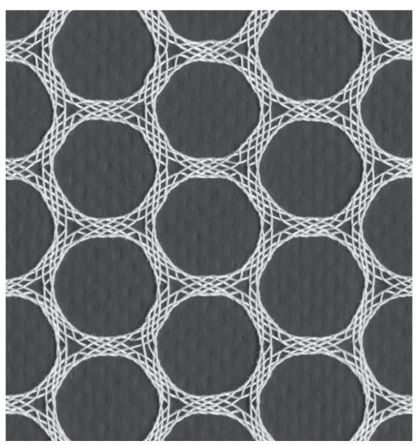
TOM EDGAR (MR Author ID: 821633) is an Associate Professor of Mathematics and first became interested in Abel's transformation due to its relationship to Möbius inversion on posets.

Artist Spotlight: Veronika Irvine



Bee prepared, Veronika Irvine; White cotton thread, 2015. This pattern has *632 orbifold symmetry. The honeycomb structure is formed on a hexagonal lattice.

See interview on page 307–309.