1.2 Chord Tables of Hipparchus and Ptolemy

Hipparchus:

The birth of trigonometry occurred in the chord tables of Hipparchus (c 190 - 120 BCE) who was born shortly after Eratosthenes died. Hipparchus introduced the full Babylonian sexigesimal notation for numbers including the measurement of angles using degrees, minutes, and seconds into Greek science. Hipparchus applied Babylonian computational methods in astronomy as well as geography. It was the Babylonians who first used the idea of coordinates to describe the location of stars in sky. Hipparchus modified their scheme by measuring positions by degrees along (and then at right angles to) the celestial equator rather than the ecliptic (circular path of the sun through the stars).

The Babylonian computational methods turned the descriptive cycles and epicycles used in the astronomy of Eudoxus and Apollonius into a true computational tool. One of the key ingredients in this astronomy was a table of chords Hipparchus constructed that enabled him to “solve” (i.e. find all the angles and sides of) right triangles that come up in astronomical problems in the same way that we use trigonometry. His tables are lost, but almost surely consisted of lengths of chords in a standard circle for all angles from 0 to 180 degrees in steps of $7\frac{1}{2}$ degrees [Toomer, 19xx].

We can only conjecture how he may have originally thought of constructing a table of chords. We know his table was used for problems in astronomy, but there is another problem that would have been of interest to Hipparchus that illustrates the usefulness of a chord table.

Like Eratosthenes, Hipparchus also wrote a work in (at least) three books on geography which he titled: *Against the Geography of Eratosthenes.* He pointed out several inconsistencies in the distances and data of Eratosthenes.

One problem that conceivably could have faced both men is the difficulty of converting an East-West distance between two points given in stades to a difference in longitude that could be represented on a map of the world globe. After Eratosthenes’ measurement of the earth, converting North-South distances into differences in latitude is just an exercise in proportion because the distances are along a great circle and we know its size. But East-West travel is not on a great circle unless one is at the equator. One must first find the size of the circle, which depends on latitude, before the longitude problem can be solved.

Suppose you travel due east until you circumnavigate the globe and arrive back at your starting point. How far do you travel? Clearly the answer is different for those who live in Malindi, Kenya on the equator and those who live in

Salem, Oregon on the 45th parallel. The Kenyans travel on a great circle, while the Oregonians have a shorter journey. Santa Claus has an afternoon stroll.

The sketch below shows a circle of constant latitude seen from the top view (looking straight down at the pole) and side view (looking at the edge of the equatorial plane.

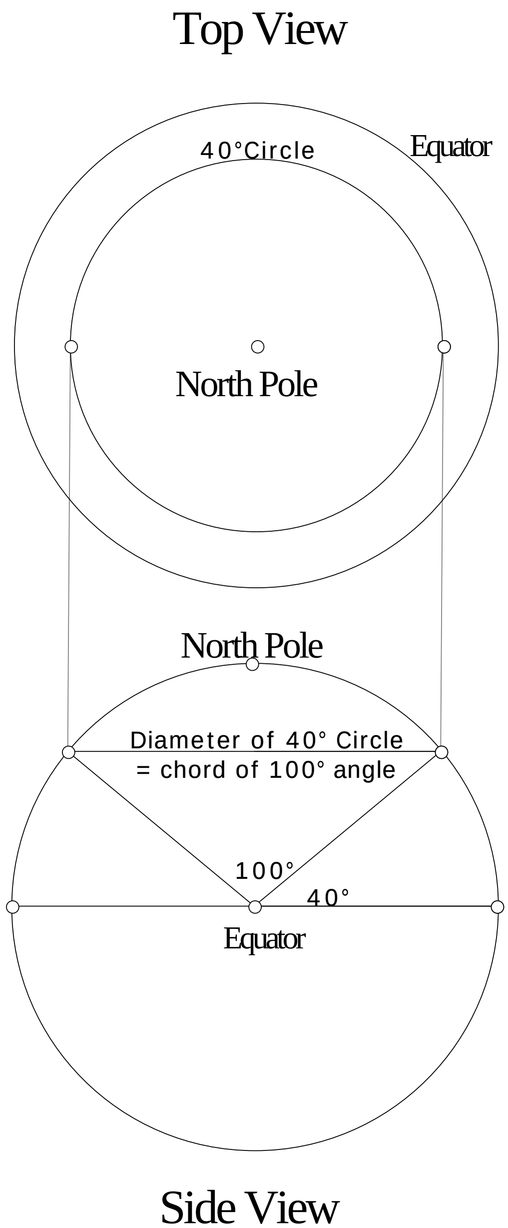

Figure 1: Chords and Latitudes

In the figure the constant lattitude is L=40^o. In general, the diameter of a circle of constant latitude, L, is the chord of the angle: 180-2L. Once the diameter of the circle is known, the circumference can be found and lengths of arcs converted into degrees. Thus a table of chords would provide the means to convert East-West distances into changes in longitude at various latitudes. Presumably one could interpolate to convert East-West distances that occurred at latitudes that fell between those handled by the table.

Toomer, 19xx points out that Hipparchus likely used a procedure that corresponds to a "Half-Angle" formula for chords that is analogous to our "Half-Angle" formulas for the sine and cosine and that he had no similar method corresponding to an "Angle-Sum" formula. That would have to wait for Ptolemy. Katz, 1998 also describes the method Toomer suggests for Hipparchus.

Exercises:

1. Hipparchus used a circle of radius 3438 for his chord table. Find the circumference of the circle.

2. The circumference (rounded off) should remind you of a number in the exercise set on Eratosthenes. Which one? How is it related to the Babylonian system of degrees, minutes, etc. ?

3. Use the method described on p. 144 of Katz (our text) to find the chord of 30 degrees, starting with the known value of the chord of 60 degrees.

Ptolemy:

In the middle of the second century of our era Claudius Ptolemais, Ptolemy, published his work on astronomy which we know as the Almagest since Islamic writers referred to it as al-magisti: the greatest. It included a chord table for angles from 0 to 180 degrees in steps of $\frac{1}{2}$ degree. His work was so superior to any previous one that the others fell into disuse until only the Almagest remained. Science's gain history's loss. Ptolemy used a theorem on cyclic quadrilaterals (now called Ptolemy's Theorem) in much the same way we use the addition formulas for the sine and cosine in order to produce his table.

Aaboe, 1964 and Berggren, 1986, and Katz, 1998 all contain excellent explanations of how Ptolemy approximated the sine of one-half degree and then used his theorem to construct the table. They also explain how tables of chords can be used to find all the angles and sides of a right triangle. All three books are excellent resources and contain exercises.

The original chord tables of trigonometry are hardly mentioned in modern texts. At least one reason is that the geometric methods and presentation appear as remote in style from our modern treatment as they are in time. The following discussion of how Ptolemy used his theorem to create his table is based on the sources mentioned above, but introduces one further bit of notation that illuminates the similarities between Greek trigonometry and modern trigonometry.

Modern writers abbreviate the chord of angle u as crd(u). Let us also introduce the notion of the supplementary chord of u, abbreviated scd(u), defined as the chord of the angle supplementary to u. Note the similarity to the cosine as the sine of the complementary angle. The relation between crd(u) and scd(u) is shown in Figure 3.

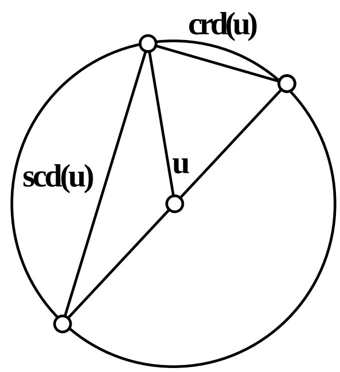

Figure 3: scd(u) the supplementary chord of u

The first analogy between Greek and modern trigonometry comes from noting that, since the triangle of Figure 3 is inscribed in a semi-circle, it is a right triangle. Hence we may apply the Pythagorean theorem. If D is the diameter of the circle, we get what we will call the Pythagorean Chord Identity:

$$\mathrm{crd}^2(u) + \mathrm{scd}^2(u) = D^2.$$

Once one fixes a value for D, scd(u) can be computed from crd(u) and vice versa. Note the striking resemblance of this relation to the relation between the sine and cosine, especially if one were to choose the value of D to be 1.

Exercise:

4. Ptolemy used a circle of radius 60 for his chord table. Find crd(60°) and then use the above relation to find crd(120°).

The next analogy comes from Ptolemy's Theorem:

> If ABCD is a quadrilateral inscribed in a circle, then the sum of the products of the lengths of the opposite sides is equal to the product of the length of the diagonals.

If Cabri or Geometer's Sketchpad is available, this relation can be discovered experimentally (See Figure 4).

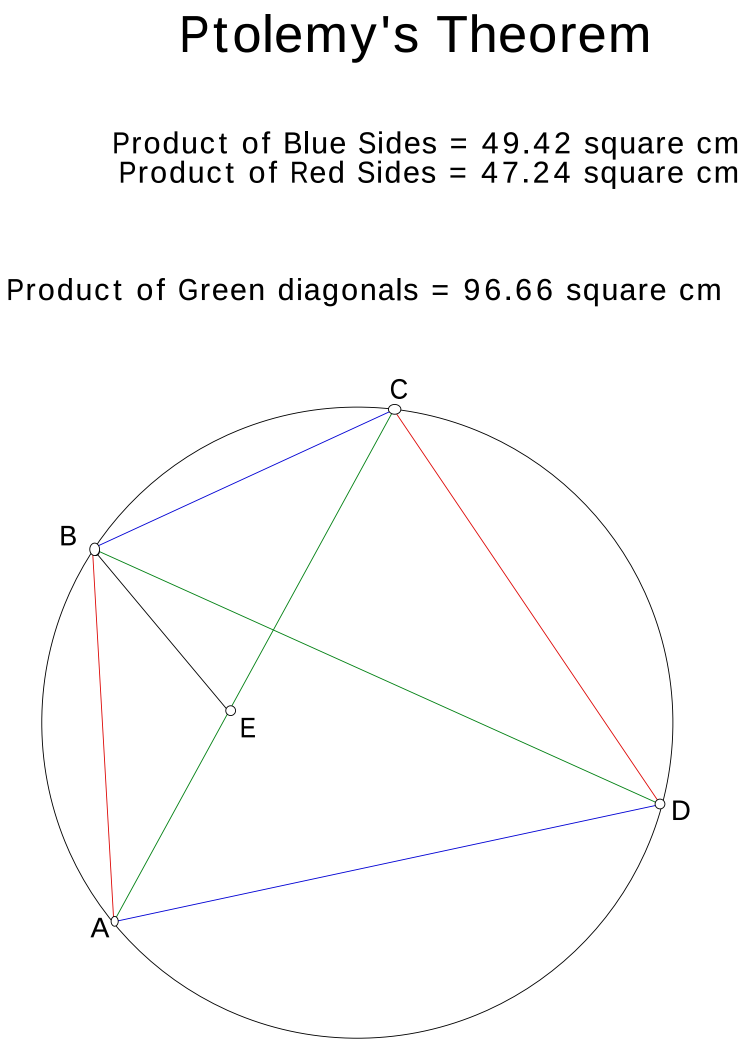

Figure 4: Ptolemy’s Theorem
(The top and bottom sides are blue and the left and right sides are red.)

Proofs of Ptolemy’s Theorem can be found in Aaboe, 1964, Berggren, 1986, and Katz, 1998.

Exercise:

5. Verify Ptolemy's theorem using Geometer's Sketchpad or Cabri.
6. Draw BE so that angle ABE has same measure as angle CBD. Make a list of all the similar triangles you can find in the figure. (Hint: remember the theorems about sizes of angles and arcs.)
7. Can you see how to use your similar triangles to prove Ptolemy's theorem? See our text if you get stuck.

How does Ptolemy's Theorem give a formula for crd(u+v) in terms of crd(u), crd(v) and the corresponding supplementary chords? Figure 5 shows what we are trying to find.

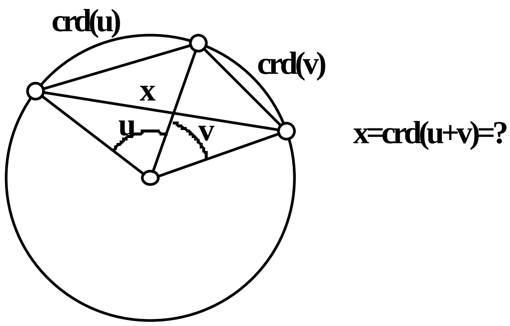

Figure 5: The chord of the sum of two angles

By extending the common side of angles u and v to a diameter of length D, we obtain the cyclic quadrilateral shown in Figure 6.

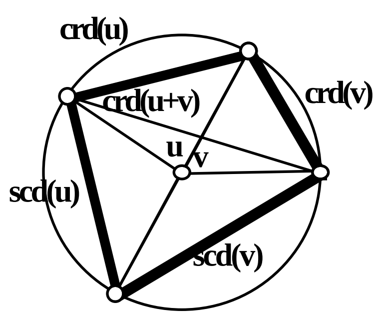

Figure 6

Applying Ptolemy's Theorem to this quadrilateral we get the angle sum identity:

$$crd(u+v)\ D = crd(u)\ scd(v)+crd(v)scd(u).$$

Again we are reminded of an old friend:

$$\sin(u+v)=\sin(u)\cos(v)+\sin(v)\cos(u),$$

especially if we take D=1. Starting with his approximation of crd($\frac{1}{2}^{o}$), Ptolemy could proceed as follows: use the Pythagorean Identity above to find scd($\frac{1}{2}^{o}$), then use the angle sum identity can be used to find crd(1^{o}) = crd($\frac{1}{2}^{o} + \frac{1}{2}^{o}$), then the Pythagorean Identity to find scd(1^{o}) and proceeding in this fashion complete his table.

In fact neither Ptolemy nor Hipparchus used a circle of diameter 1. Hipparchus apparently used a circle of radius 3438 so that the measure of the circumference would be (nearly) equal to the number of minutes in 360^{o}. (It's fun to find the value of π implicit in Hipparchus' choice of radius.) Ptolemy used a circle of radius 60, presumably because 60 is the base of the Babylonian number system which he used for computation.

Thus the infusion of the Babylonian sexagesimal arithmetic from ancient Iraq into the geometric mathematics of Greece culminated in the Almagest, which was the ultimate authority regarding astronomy in the West for over 1000 years. It also produced the chord tables of Hipparchus which spawned the sine function upon their arrival in India.